

ENERGY THEOREMS AND CRITICAL LOAD APPROXIMATIONS IN THE GENERAL THEORY OF ELASTIC STABILITY*

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1. Introduction. When the ordinary uniform pinned-end column buckles under a critical load $P_1 = \pi^2 EI/l^2$, the potential energy measured from the straight compressed form is

$$V = \frac{1}{2} EI \int_0^l y''^2 dx - \frac{1}{2} P_1 \int_0^l y'^2 dx$$

or

$$\frac{2V}{EI} = \int_0^l y''^2 dx - \frac{\pi^2}{l^2} \int_0^l y'^2 dx \quad (1)$$

and is zero since $Y \propto \sin \pi x/l$.

Let y now be a deflection of any form which satisfies the end conditions. Then V

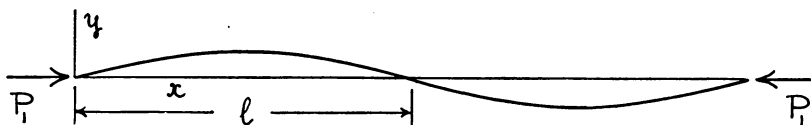


FIG. 1.

as given by (1) is *positive* unless $y \propto \sin \pi x/l$, in which case it is zero. This follows readily from a Fourier expansion¹ of y , or from Wirtinger's inequality,² which is that

$$\int_0^{2\pi} z^2 dx < \int_0^{2\pi} z'^2 dx, \quad (2)$$

unless $z = A \cos x + B \sin x$, provided that

$$z(0) = z(2\pi) \quad \text{and} \quad \int_0^{2\pi} z dx = 0.$$

For if we reflect the bar with its deflection y in its right hand end ($x = l$) and combine the inverted reflection with the original bar (Fig. 1), we have a bar of length $2l$, with $y'(0) = y'(2l)$, and from (1)

$$\frac{4V}{EI} = \int_0^{2l} y''^2 dx - \frac{\pi^2}{l^2} \int_0^{2l} y'^2 dx.$$

*Received February 5, 1951.

¹See for instance R. V. Southwell, *An introduction to the theory of elasticity for engineers and physicists*, Oxford University Press, 1941, p. 444.

²G. H. Hardy, E. H. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1934, p. 185. The authors are indebted to Prof. Polya for the suggestion that this inequality might serve the purpose.

The substitution $\xi = \pi x/l$ converts this to

$$\frac{4V}{EI} = \left(\frac{\pi}{l}\right)^3 \left[\int_0^{2\pi} y''^2 d\xi - \int_0^{2\pi} y'^2 d\xi \right], \quad (3)$$

where the prime now means differentiation with respect to ξ , and if we identify y' with z , ξ with x in (2), (2) establishes that V is positive unless y is sinusoidal.

Two conclusions follow. *First*, that since V is the potential energy of the bar under the true critical load P_1 , the sinusoidal buckled form is itself stable with respect to disturbances (impulses) which project it into non-sinusoidal forms. It is neutral with respect to sinusoidal disturbances. In the disturbances P_1 remains unchanged. *Second*, that an approximation P_a calculated by inserting an assumed approximate deflection y_a for y in the relation

$$\frac{1}{2} EI \int_0^l y''^2 dx - \frac{1}{2} P \int_0^l y'^2 dx = 0 \quad (4)$$

(valid when P is P_1 and y is sinusoidal) will be higher than P_1 . For the use of (4) in this way will yield

$$P_a = \frac{EI \int_0^l y_a''^2 dx}{\int_0^l y_a'^2 dx}. \quad (5)$$

But since y_a is not sinusoidal we have $V > 0$ and (1) yields

$$P_1 < \frac{EI \int_0^l y_a''^2 dx}{\int_0^l y_a'^2 dx}. \quad (6)$$

Here we have an inequality (2) available from pure mathematics, and can use it to establish either the first or the second conclusion.

In a plate problem the corresponding inequality, establishing that the potential energy, (strain energy of bending minus work of critical loads on buckling displacements), is positive for any displacement differing from the true buckling displacement, is a much more elaborate one, although it can of course be formulated. A proof by means of Fourier series is feasible for the simpler cases, such as the rectangular plate with four simply-supported edges, but a proof for the more difficult cases such as four clamped edges is hardly to be expected. Still less can we hope to obtain such a proof for more complex systems such as shells, or combinations of structural elements such as stiffened plates and shells, or the general problem of elastic stability with respect to infinitesimal displacements.

But if we are *given* that the buckled form is itself not unstable, this datum establishes the inequality, and we can then use it to prove that the energy approximations to the critical loads will be too high. In the remainder of the paper we do this for the general stability problem. If the buckled state *were* itself unstable, the energy approximation to the critical load would be too low. Thus the usual assumption in practical calculations that the approximation will be too high is equivalent to the assumption that in the idealized version of the problem there *is* a buckled state which is itself not unstable.

2. Formulation of the general equations. An arbitrary elastic solid has initial stress specified by the usual Cartesian components S_{ij} ($i, j = 1, 2, 3$), which maintain equilibrium with initial body force F_i per unit volume and surface force T_i per unit area on a surface element whose outward direction cosines are ν_i . We have then the differential equations of equilibrium (with the summation convention for repeated indices, and subscripts after a comma indicating differentiation with respect to the corresponding coordinates)

$$S_{ii,i} + F_i = 0 \quad (7)$$

and the boundary conditions of equilibrium

$$S_{ij}\nu_j = T_i. \quad (8)$$

The stress S_{ij} is not necessarily entirely due to F_i and T_i . It may be initial or thermal stress existing in the absence of F_i and T_i .

This state of stress will be referred to as state I, and x_i are the co-ordinates of material points in this state (not in the unstressed state). For the present, we suppose that it is stable. A second state, state II, is derived from it by the application of additional body force ΔF_i and additional surface force ΔT_i . The displacement caused is expressed by Cartesian components u_i (not Δu_i), and it is affected by the presence of the initial stress. The stress in state II is of course different from S_{ij} . To specify it we use Trefftz's stress components³ k_{ij} (in Kappus' notation). These are non-orthogonal. A rectangular block element in state I becomes an elementary parallelepiped in state II, and these stress components refer to the directions of its edges. The advantage of using them is that they lead to relatively simple equations. We may write

$$k_{ij} = S_{ij} + \tau_{ij} \quad (9)$$

and $\tau_{ij} = \tau_{ji}$ since both $S_{ij} = S_{ji}$ and $k_{ij} = k_{ji}$. Even where the τ_{ij} vanish, this state of stress need not be identical with that expressed by the S_{ij} of state I, on account of the different specification of stress components.

The differential equations of equilibrium⁴ satisfied by τ_{ij} are

$$\tau_{ii,i} + (S_{ik}u_{i,k})_{,i} + \Delta F_i = 0 \quad (10)$$

after neglecting "non-linear" terms $(\tau_{ik}u_{i,k})_{,i}$, and so restricting the investigation to τ_{ij} small compared with S_{ij} —more precisely to the largest τ_{ij} small compared with the largest S_{ij} .

The boundary conditions of equilibrium satisfied by τ_{ij} are

$$\tau_{ij}\nu_j + S_{ik}u_{i,k}\nu_i = \Delta T_i. \quad (11)$$

When $S_{ij} = 0$, (10) and (11) reduce as they should to the equations of the ordinary theory of elasticity.

Equations (10) and (11) do not involve any stress-strain relations. Being concerned with small departures from state I, we assume that small strain components

$$e_{ii} = \frac{1}{2}(u_{i,i} + u_{i,i}) \quad (12)$$

³E. Trefftz, *Zur Theorie der Stabilität des elastischen Gleichgewichts*, Z. angew. Math. Mech. 12, 160 (1933).

⁴See the reference in footnote (3).

are related to the small stress components τ_{ij} by the usual form of Hooke's Law. It is convenient to take this in the general form appropriate to the homogeneous anisotropic solid, and to express this form in a changed notation. Write $\tau_{11} = \tau_1$, $\tau_{22} = \tau_2$, $\tau_{33} = \tau_3$, $\tau_{12} = \tau_4$, $\tau_{23} = \tau_5$, $\tau_{31} = \tau_6$ and similarly for the strain components. Then the stress-strain relations are⁵

$$\tau_i = c_{ij}e_j, \text{ where } i \text{ and } j \text{ have the range 1 to 6, and } c_{ij} = c_{ji} \quad (13)$$

The variational principles (stationary potential energy for equilibrium, Castigliano's Theorem, etc.) of the ordinary theory of elasticity can be derived by considering the variation of the strain energy of the body, and using the equations of equilibrium.⁶ We follow this method now for the transition from state I to state II, with some modification, using the equations of this article. We thus regard equations of equilibrium (or motion) as basic, and energy principles such as stationary potential energy as derived, rather than vice-versa.

3. A variational principle. From the quantities e_i ($i = 1$ to 6) as functions of the co-ordinates x_i ($i = 1$ to 3) in state I we may form by integration over all the volume elements $d\omega$ of state I the integral

$$U(e) = \frac{1}{2} \int c_{ij}e_ie_j d\omega \quad (14)$$

which would be the strain energy in the absence of initial stress. Trefftz⁷ has shown that the strain energy acquired in the passage from state I to state II is

$$U(e) + \int S_{ij}e_{ij} d\omega + \frac{1}{2} \int S_{ik}u_{i,i}u_{i,k} d\omega, \quad (i, j = 1, 2, 3). \quad (15)$$

We consider the variation of $U(e)$ alone. Let arbitrary variations δu_i be added to the displacements u_i . Then, writing δU for the complete variation of U we have from (14)

$$\begin{aligned} U(e) + \delta U &= \frac{1}{2} \int c_{ij}(e_i + \delta e_i)(e_j + \delta e_j) \quad (i, j = 1 \text{ to } 6) \\ &= U(e) + \frac{1}{2} \int c_{ij}(e_i \delta e_j + e_j \delta e_i) d\omega + U(\delta e), \end{aligned}$$

where

$$U(\delta e) = \frac{1}{2} \int c_{ij} \delta e_i \delta e_j d\omega. \quad (16)$$

Since $c_{ij} = c_{ji}$, we have

$$\delta U = \int c_{ij}e_i \delta e_j d\omega + U(\delta e) = \int \tau_j \delta e_j d\omega + U(\delta e). \quad (17)$$

We now return to the range 1, 2, 3 for i, j and k , and write instead of (17):

$$\delta U = \int \tau_{ii} \delta e_{ii} d\omega + U(\delta e). \quad (18)$$

⁵A. E. H. Love, *Mathematical theory of elasticity*, Cambridge University Press, 1927, Ch. 3.

⁶E. Trefftz, *Handbuch der Physik*, Vol. 6, Springer, Berlin, 1928, p. 68.

⁷See the reference in footnote (3).

For the integral in (18) we have, taking δe_{ij} from (12),

$$\begin{aligned} \frac{1}{2} \int \tau_{ij} (\delta u_{i,j} + \delta u_{j,i}) d\omega &= \int \tau_{ij} \delta u_{i,j} d\omega \quad (\text{since } \tau_{ij} = \tau_{ji}) \\ &= \int (\tau_{ij} \delta u_i)_{,j} d\omega - \int \tau_{ij,i} \delta u_i d\omega. \end{aligned}$$

By an application of the divergence theorem to the first of these two integrals we may write the result as

$$\int_{\Sigma} \tau_{ij} \delta u_i \nu_j d\sigma - \int \tau_{ij,i} \delta u_i d\omega,$$

where $d\sigma$ and Σ refer to the boundary surface and ν_j are the direction cosines of the normal, all in state I. We now eliminate τ_{ij} by use of the boundary conditions (11) in the first integral and the equilibrium equations (10) in the second. The result is

$$\int_{\Sigma} [-S_{ik} u_{i,k} \nu_j + \Delta T_i] \delta u_i d\sigma + \int [(S_{ik} u_{i,k})_{,i} + \Delta F_i] \delta u_i d\omega \quad (19)$$

The first part of the second integral is transformed as follows

$$\begin{aligned} \int (S_{ik} u_{i,k})_{,i} \delta u_i d\omega &= \int (S_{ik} u_{i,k} \delta u_i)_{,i} d\omega - \int S_{ik} u_{i,i} \delta u_{i,k} d\omega \\ &= \int_{\Sigma} S_{ik} u_{i,k} \delta u_i \nu_i d\sigma - \int S_{ik} u_{i,i} \delta u_{i,k} d\omega. \end{aligned}$$

With this (19) is simplified by cancellation of the two surface integrals involving S_{ik} . Recalling that (19) is equivalent to the integral in (18), we have as a new version of the latter equation

$$\delta U = \int_{\Sigma} \Delta T_i \delta u_i d\sigma + \int \Delta F_i \delta u_i d\omega - \int S_{ik} u_{i,i} \delta u_{i,k} d\omega + U(\delta e). \quad (20)$$

In this the u_i are the actual displacements caused by the application of the additional forces ΔT_i and ΔF_i , corresponding to the passage from state I to state II, and the δu_i are arbitrary additional displacements. Both the u_i and the δu_i are restricted to smallness by (12) and (13).

Now let S_{ij} , F_i , T_i , ΔF_i , ΔT_i be fixed, but let u_i for the moment be three independent functions of the x_i , not required as yet to be the correct displacements in the passage from state I to state II. The result (20) suggests consideration of a function of these u_i in the form

$$V = U(e) - \int_{\Sigma} \Delta T_i u_i d\sigma - \int \Delta F_i u_i d\omega + \frac{1}{2} \int S_{ik} u_{i,i} u_{i,k} d\omega. \quad (21)$$

On varying the u_i (as they appear explicitly, and also in $U(e)$) we have

$$\delta V = \delta U - \int_{\Sigma} \Delta T_i \delta u_i d\sigma - \int \Delta F_i \delta u_i d\omega + \frac{1}{2} \int S_{ik} \delta(u_{i,i} u_{i,k}) d\omega. \quad (22)$$

Now let u_i be given the actual values of state II. Then on account of (20) we have

$$\delta V = \int S_{jk} \left[\frac{1}{2} \delta(u_{i,j} u_{i,k}) - u_{i,j} \delta u_{i,k} \right] d\omega + U(\delta e). \quad (23)$$

But

$$\begin{aligned} \frac{1}{2} S_{jk} \delta(u_{i,j} u_{i,k}) &= \frac{1}{2} S_{jk} [(u_i + \delta u_i)_{,j} (u_i + \delta u_i)_{,k} - u_{i,j} u_{i,k}] \\ &= \frac{1}{2} S_{jk} [\delta u_{i,j} u_{i,k} + u_{i,j} \delta u_{i,k} + \delta u_{i,j} \delta u_{i,k}]. \end{aligned}$$

Since $S_{jk} = S_{kj}$ the first term in the brackets can be combined with the second to give the result

$$\frac{1}{2} S_{jk} [2u_{i,j} \delta u_{i,k} + \delta u_{i,j} \delta u_{i,k}].$$

Then (23) reduces to

$$\delta V = 0 + U(\delta e) + \frac{1}{2} \int S_{jk} \delta u_{i,j} \delta u_{i,k} d\omega, \quad (24)$$

the zero indicating that the first order (in δu_i) variation of V vanishes. This property of course would be characteristic of the potential energy in state II as an equilibrium state. Referred to state I as zero, the potential energy consists of the strain energy (15) together with the potential energy of the body and surface forces, which is given for state II by

$$- \int_{\Sigma} (T_i + \Delta T_i) u_i d\sigma - \int (F_i + \Delta F_i) u_i d\omega.$$

It can be shown by means of (7) and (8) that the terms here in T_i and F_i cancel the middle term in (15), and hence that V as given by (21) is in fact the potential energy of state II when the u_i in (21) denote the actual displacements of state II.

4. The stability of state II. Our object being to deduce a generalization of the inequality (6) when the stability of state II is given, we now seek a necessary condition for this stability.

Let the particles of the body be projected from state II by some disturbance. Then at time t they are in motion with displacements δu_i (functions of the co-ordinates x_i of the particles in state I), and corresponding to δu_i we have additions $\delta \tau_{ij}$ to the Trefftz stress components (9). There is, by hypothesis, no change in the forces $F_i + \Delta F_i$, $T_i + \Delta T_i$ of state II except at fixed supports where reactions may be induced. During the motion these are carried with the particles on which they act in state II. The equations of motion are, from (10)

$$(\tau_{ij} + \delta \tau_{ij})_{,j} + [S_{jk}(u_i + \delta u_i)_{,k}]_{,j} + \Delta F_i = \rho \delta \ddot{u}_i,$$

where $\delta \ddot{u}_i = \partial^2 \delta u_i / \partial t^2$ (the acceleration) and ρ is the density. Subtracting (10) we have

$$\tau_{ij,j} + (S_{jk} \delta u_{i,k})_{,j} = \rho \delta \ddot{u}_i.$$

Multiplying by $\delta \dot{u}_i$ and integrating over the volume we find

$$\int \tau_{ij,j} \delta \dot{u}_i d\omega + \int (S_{jk} \delta u_{i,k})_{,j} \delta \dot{u}_i d\omega = \frac{d}{dt} \int \frac{1}{2} \rho \delta \dot{u}_i \delta \dot{u}_i d\omega, \quad (25)$$

and the term on the right is the time derivative of the kinetic energy. The first integral on the left of (25) can be written as

$$\int (\delta\tau_{ij} \delta\dot{u}_i)_{,i} d\omega - \int \delta\tau_{ij} \delta\dot{u}_{i,i} d\omega$$

and after transformation of the first of these integrals by the divergence theorem, as

$$\int_{\Sigma} \delta\tau_{ij} \delta\dot{u}_i \nu_j d\sigma - \int \delta\tau_{ij} \delta\dot{u}_{i,i} d\omega.$$

Similarly the second integral on the left of (25) transforms into

$$\int_{\Sigma} S_{jk} \delta u_{i,k} \delta\dot{u}_i \nu_j d\sigma - \int S_{jk} \delta u_{i,k} \delta\dot{u}_{i,i} d\omega.$$

Introducing these transformations in (25), and writing T for the kinetic energy, we have, with some rearrangement,

$$\frac{dT}{dt} + \int (\delta\tau_{ij} + S_{jk} \delta u_{i,k}) \delta\dot{u}_{i,i} d\omega = \int_{\Sigma} (\delta\tau_{ij} + S_{jk} \delta u_{i,k}) \delta\dot{u}_i \nu_j d\sigma. \quad (26)$$

The bracket appearing in the integral on the right is, by (11), the addition to ΔT_i accompanying the motion, and this is zero by hypothesis except at fixed supports, where the $\delta\dot{u}_i$ vanish. The integral therefore vanishes. The integral on the left of (26) is the same as

$$\frac{d}{dt} \left\{ \frac{1}{2} \int \delta\tau_{ij} \delta e_{ij} d\omega + \frac{1}{2} \int S_{jk} \delta u_{i,i} \delta u_{i,k} d\omega \right\}. \quad (27)$$

This is readily verified by changing $\delta\tau_{ij}\delta e_{ij}$ to $\delta\tau_i\delta e_i$ ($i = 1$ to 6), then to $c_{ij}\delta e_i\delta e_j$, carrying out the differentiation with respect to t , and making the combinations of terms permitted by $c_{ij} = c_{ji}$ and $S_{jk} = S_{kj}$. It is evident that the bracket in (27) is identical with δV as given by (24). We can therefore re-express (26) as

$$\frac{dT}{dt} + \frac{d\delta V}{dt} = 0 \quad (28)$$

showing that $T + \delta V$ remains constant during the motion following the projection from the equilibrium state II. This of course is the energy equation of this motion, and exhibits δV as the potential energy referred to state II under the conditions of this motion—no change of body and surface force except at fixed supports.

Stability of state II implies an immediate decrease in the kinetic energy following the projection from state II, and therefore an immediate increase of δV . Thus stability means that δV as given by (24) is positive for arbitrary δu_i . If it is given that state II is not unstable, (24) is not negative.

We may take the u_i to be zero, as a special case, state II then being the same as state I. Evidently the equilibrium in state I, under the initial stress S_{ij} , will be unstable when the right hand side of (24) is negative for any δu_i which vanish at fixed supports.

The value which V , as given by (21), takes when the u_i are the correct displacements

of state II can be reduced to a simpler and useful form. Writing $-I$ for the last integral in (21) we have

$$\begin{aligned} -I &= \frac{1}{2} \int S_{ik} u_{i,j} u_{i,k} d\omega \\ &= \frac{1}{2} \int (S_{ik} u_{i,k})_{,i} d\omega - \frac{1}{2} \int (S_{ik} u_{i,k})_{,i} u_i d\omega \\ &= \frac{1}{2} \int_{\Sigma} S_{ik} u_{i,k} u_{i,j} d\sigma - \frac{1}{2} \int (S_{ik} u_{i,k})_{,i} u_i d\omega. \end{aligned}$$

Using (11) and (10) respectively in the first and second of these integrals we find

$$-I = \frac{1}{2} \int_{\Sigma} \Delta T_i u_i d\sigma - \frac{1}{2} \int_{\Sigma} \tau_{ij} u_{i,j} d\sigma + \frac{1}{2} \int \Delta F_i u_i d\omega + \frac{1}{2} \int \tau_{ij,i} u_i d\omega \quad (29)$$

If in the last integral we write

$$\tau_{ij,i} u_i = (\tau_{ij} u_i)_{,i} - \tau_{ij} u_{i,i},$$

the first of the two resulting integrals will, by the divergence theorem, cancel the second integral on the right of (29). Then, observing that, since $\tau_{ij} = \tau_{ji}$ and e_{ij} is given by (12),

$$\frac{1}{2} \int \tau_{ij} u_{i,j} d\omega = \frac{1}{2} \int \tau_{ij} e_{ij} d\omega = U(e),$$

we can rewrite (29) as

$$-I = \frac{1}{2} \int_{\Sigma} \Delta T_i u_i d\sigma + \frac{1}{2} \int \Delta F_i u_i d\omega - U(e).$$

This form is valid only when u_i are the correct displacements of state II, because (10) and (11) have been incorporated. Returning to (21) we have the corresponding value of V as

$$V = -\frac{1}{2} \int_{\Sigma} \Delta T_i u_i d\sigma - \frac{1}{2} \int \Delta F_i u_i d\omega \quad (30)$$

5. Elastic buckling. We have so far been concerned with two neighboring equilibrium states, state I and state II, the passage from state I to state II being effected by additional surface and body forces ΔT_i , ΔF_i . When the passage is a buckling deformation, there will be no change in body force (e.g. gravity), but for very exceptional problems, and we may take $\Delta F_i = 0$. The surface forces may be taken to change at supports (e.g. the transverse reactions induced when buckling occurs in a column with one end clamped, the other pinned), but not elsewhere (the loads remain unchanged during buckling, moving with the particles they act on).

Let the supports be such that no work is done by the reactions on the buckling displacements (as is true for the common boundary conditions of bars and plates. If work is done, as by elastic restraining moments, the elastic restraints may be included in the structure, and *their* fixed supports are then of the assumed type).

Then each integral in (30) vanishes, and therefore $V = 0$. Thus (21) now yields

$$V = U(e) + \frac{1}{2} \int S_{ik} u_{i,j} u_{i,k} d\omega = 0 \quad (31)$$

which is a generalized energy relation valid when S_{ik} is a critical state of stress, and the u_i are actual buckling displacements.

But $U(e)$ is necessarily positive for any e_{ij} since it has the form of the strain energy in the absence of initial stress. We have therefore from (31)

$$-I = \frac{1}{2} \int S_{ik} u_{i,j} u_{i,k} d\omega < 0 \quad (32)$$

and hence for a critical state I is positive.

The initial stress S_{ik} , now of course a critical state of stress under which buckling from state I to state II is possible, can be represented as

$$S_{ik} = \Gamma S_{ik}^0, \quad (33)$$

where S_{ik}^0 is a non-critical stress tensor having the same distribution but a non-critical magnitude, and Γ is a positive multiplier. The S_{ik}^0 being chosen, we inquire what value of Γ corresponds to a critical state. It now follows from (32) that

$$I^0 = -\frac{1}{2} \int S_{ik}^0 u_{i,j} u_{i,k} d\omega > 0 \quad (34)$$

which defines I^0 .

Introducing (33) in (31), and using the equality in (34) we have

$$\Gamma = \frac{U(e)}{I^0}. \quad (35)$$

This holds, giving the critical value of Γ , when $U(e)$ and I^0 are evaluated from the correct buckling displacements u_i .

We now write Γ' for the quantity which is calculated from the formula (35) using functions u'_i other than u_i in $U(e)$ and I^0 , leaving S_{ik}^0 in the latter unchanged. We can then inquire what choice of u'_i will yield the least value of Γ' . Let $u'_i = u_i + \delta u_i$, u_i being the correct buckling displacements, and correspondingly write $\Gamma' = \Gamma + \delta\Gamma$. Then

$$\Gamma + \delta\Gamma = \frac{U(e) + \delta U}{I^0 + \delta I^0} = \frac{U(e) + \Gamma \delta I^0 + (\delta U - \Gamma \delta I^0)}{I^0 + \delta I^0}. \quad (36)$$

Let the variations δu_i satisfy the same boundary conditions as the u_i . Then

$$\int_{\Sigma} \Delta T_i \delta u_i d\sigma = 0,$$

and we have from (22)

$$\begin{aligned} \delta V &= \delta U + \frac{1}{2} \int S_{ik} \delta(u_{i,j} u_{i,k}) d\omega \\ &= \delta U + \frac{1}{2} \Gamma \int S_{ik}^0 \delta(u_{i,j} u_{i,k}) d\omega \\ &= \delta U - \Gamma \delta I^0 \end{aligned}$$

by the definition of I^0 in (34). Putting this in the last member of (36), and at the same time replacing $U(e)$ by ΓI^0 from (35), we find

$$\delta\Gamma = \frac{\delta V}{I^0 + \delta I^0}. \quad (37)$$

In section 4 we found that when state II is itself stable δV as given by (24) is positive. We now define state II as a buckled state which is itself stable with respect to disturbances which project it into a different configuration (the $u_i + \delta u_i$ not merely proportional to the u_i). These disturbances correspond to the "non-sinusoidal disturbances" of the column in section 1. Our "incorrect" displacements $u_i + \delta u_i$ are of this character, and therefore δV is positive. Since, by (24), δV is zero in the first order (in δu_i) and positive in the second, we have from (37) that $\delta\Gamma$ is also zero in the first order, positive in the second order, and hence that the correct value Γ is the lower bound of the approximations Γ' .

Unlike δV , $\delta\Gamma$ does not terminate with the terms of the second order. It is conceivable that the δu_i could be chosen (not small compared with the u_i) so that the denominator in (37) becomes negative. Then the approximation Γ' would be *smaller* than Γ .

Acknowledgement. The investigation reported in this paper was supported by the Office of Naval Research through a contract (N6 ONR T.O. 12) with Stanford University.