AN INTEGRAL EQUATION APPROACH TO THE PROBLEM OF WAVE PROPAGATION OVER AN IRREGULAR SURFACE*

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In this paper we propose to outline a certain approximate technique for solving the wave equation under the following conditions. Consider a surface S which resembles somewhat the surface drawn in Fig. 1; it stretches out to infinity along a horizontal

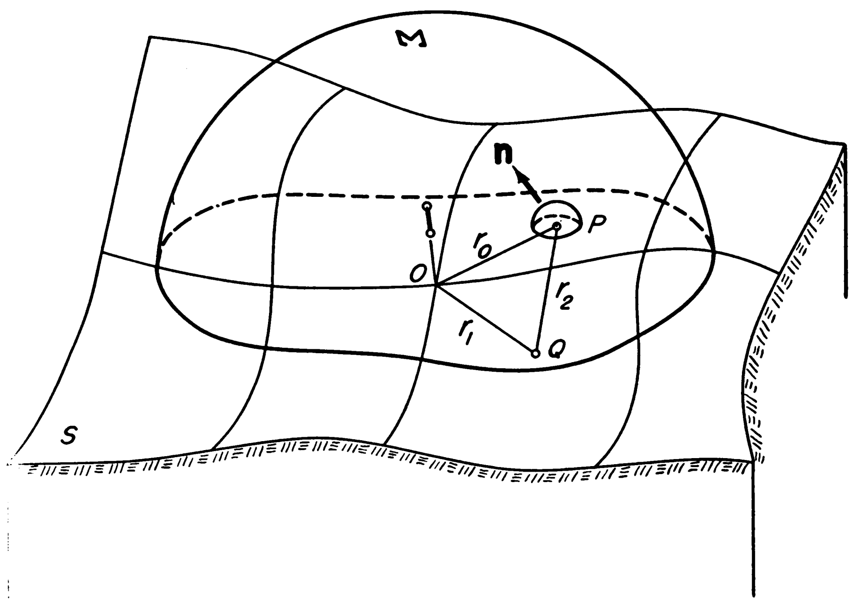

FIG. 1.

plane and is composed of "large" corrugations or bumps—large in the sense that the radius of curvature at any point of S is much larger than a wavelength. Above S there is a field $\psi\ (x,\ y,\ z)e^{-i\omega t}$ which varies harmonically in time with the radial frequency ω and which satisfies the wave equation

$$\nabla^2\psi + k^2\psi = -4\pi\tau, \tag{1}$$

k being the propagation constant ω/c and τ the distribution of sources. Finally, at the surface S the field satisfies a homogeneous boundary condition of the form

$$\frac{\partial\psi}{\partial n} = -ik\ \delta\psi, \tag{2}$$

where δ is a certain (complex) constant of proportionality.

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As we have formulated it, this problem first arose as a simplification of the problem of radio wave propagation over that irregular and inhomogeneous interface that is our Earth. At the higher frequencies being used today, radio wave transmission is greatly affected by these irregularities, and it is becoming increasingly important to find some way of estimating the resulting field strengths.

The simplifications in the present case involve (1) a scalar wave phenomenon, and (2) a homogeneous boundary condition. But, while the remaining problem is not itself without merit, it might be remarked that both assumptions can be looked upon as reasonable approximations to the original radio wave problem. The first follows from the fact that a vertically polarized source will give rise in the main to a vertically polarized field, and a horizontally polarized source to a horizontally polarized field. Thus the vertical electric field or the vertical magnetic field or a Hertz potential, all of which satisfy the scalar wave equation, will serve to characterize the field. As for the second assumption, divers arguments that lend credence to it have been presented by Schelkunoff [1], Leontovich [2], Leontovich and Fock [3], Feinberg [4], and the present author [5]. These show that in a vertically polarized field something equivalent to Eq. (2) will be satisfied with

$$\delta \approx \frac{(\eta - 1)^{1/2}}{\eta} \approx \frac{1}{n}, \tag{3}$$

where η is the complex dielectric constant of the earth

$$\eta = \frac{\epsilon}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega},$$

(ϵ/ϵ_0 being the dielectric constant and σ the conductivity) and n is the coefficient of refraction equal to $\eta^{1/2}$. Similarly in a horizontally polarized field Eq. (2) will be satisfied with

$$\delta \approx (\eta - 1)^{1/2} \approx n. \tag{4}$$

Previous authors in their attacks on the wave equation in the presence of irregular boundaries have for the most part used a perturbation method. This method was originated by Brillouin [6] and later developed by Feshbach [7] and Cabrera [8]. It begins by deriving an integral equation and then solves this by approximating the characteristic functions through known solutions to problems involving regular boundaries. However, it was pointed out by Feinberg [4] that in the special case of radio transmission this same integral equation could be reduced to a much simpler form. Subsequently, the present author in his Master's thesis [5] tried to simplify the integral equation further in the hopes that it could then be directly solved by numerical methods. The present paper is a recapitulation and a continuation of this approach.

The integral equation. According to Green's theorem, if ϕ and ψ are any two functions continuous throughout a volume V, then [9]

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dv = \int_{S^*} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) da, \tag{5}$$

where S^* is the surface bounding V. Now let us consider the volume which is bounded as in Fig. 1 by the surface S and the large hemisphere-like surface Σ. Let P be some fixed point on S and Q the variable point of integration of either the volume integral or the surface integral. Then suppose that $\psi(Q)$ satisfies Eqs. (1) and (2) while $\phi(Q)$ is the Green's function e^{ikr_2}/r_2, r_2 being the radial distance PQ. Of course, since $\phi(Q)$ as thus defined has a singularity at P, we must exclude P from the volume V by indenting it with a very small hemisphere σ.

Under these hypotheses the volume integral of Eq. (5) becomes

$$4\pi \int_V \tau(Q) \frac{e^{ikr_2}}{r_2}\, dv = 4\pi\psi_0(P), \tag{6}$$

say. This function $\psi_0(P)$ is the field that would be obtained at P if there were no earth to contend with. It is the "free-space field" due to the source distribution τ.

The surface S^* consists of three parts: Σ, σ, and S. Since ψ must satisfy the radiation condition [10], it is clear that as the radius of Σ becomes larger and larger, the integral over Σ vanishes. If the radius of σ tends to zero, then because of the singularity in $\phi(Q)$ the integral over σ approaches $2\pi\psi(P)$. Finally we know that Eq. (2) is satisfied at all points on S. Thus, after these considerations and after a little rearrangement, Eq. (5) becomes [11]

$$\psi(P) = 2\psi_0(P) + \frac{ik}{2\pi} \int_S \psi(Q) \frac{e^{ikr_2}}{r_2} \left[\delta + \left(1 + \frac{1}{kr_2}\right) \frac{\partial r_2}{\partial n} \right] da. \tag{7}$$

This as you see, is an integral equation which defines uniquely the unknown $\psi(P)$ at all points on S. We want now to simplify it with a few judicious approximations.

First, we suppose that the sources are all located on some antenna structure which is erected at a point O on S. Then we write

$$\psi_0(P) = g(P) \frac{e^{ikr_0}}{r_0}, \tag{8}$$

where r_0 is the radial distance OP. The function $g(P)$ is, in a way, the antenna pattern; it represents the gain of the antenna over an isotropic radiator at 0.

We also introduce an attentuation function $W(P)$ which is defined so that

$$\psi(P) = 2W(P) \frac{e^{ikr_0}}{r_0}. \tag{9}$$

Substituting these two equations into Eq. (7) we have

$$W(P) = g(P) + \frac{ik}{2\pi} \int_S W(Q) e^{ik(r_1 + r_2 - r_0)} \frac{r_0}{r_1 r_2} \left[\delta + \left(1 + \frac{1}{kr_2}\right) \frac{\partial r_2}{\partial n} \right] da. \tag{10}$$

Here now is the crux of the matter. Because of the factor exp $ik(r_1 + r_2 - r_0)$, the integrand in this last equation oscillates very rapidly from one point on S to another. This fact implies that we might use Kelvin's principle of stationary phases; that is, we could approximate the integral by summing up only those contributions which come from the neighborhoods of points where the phase of the integrand is stationary. These points in general form a discrete set which are exactly the "points of reflection" used in geometrical optics, and indeed, doing this should give us precisely the geometrical optics solution. It may be remarked that this very solution has already been used with some success by McPetrie and Ford [12] and by Shelleng, Burrows, and Ferrell [13] to analyze some actual data, while Keller and Keller [14] have devised several formulas to be used in this connection.

However, we cannot in our problem use this particular approximation, for when the geometrical rays are at nearly grazing angles of incidence or, even worse, when we must consider diffraction regions, then it is not nearly accurate enough. The reason for this is that at points between O and P the phase, while not necessarily stationary, is still but slowly varying. Thus to improve upon the geometrical optics solution we must reduce the integral in Eq. (10) not to the sum of discrete contributions, but to a line integral from O to P.

To do this we first project everything onto a horizontal plane S'. Then the integral can be written in the form

$$I = \int_{S'} F(Q')e^{ik(r_1'+r_2'-r_0')} \frac{da'}{r_1'r_2'},$$

where the primes are used to represent horizontal projections. Now let us construct on S' a set of elliptic coordinates (u, v) defined by

$$r_0' \cosh u = r_2' + r_1' , \qquad r_0' \cos v = r_2' - r_1' .$$

The differential area is $r_2'r_1'\ du\ dv$ and the integral becomes

$$I = \int_{-\infty}^{\infty} \exp\,[ikr_0'(\cosh u - 1)]\ du \int_0^{\pi} F(u, v)\ dv.$$

This, as an integral in u, has the phase kr_0' $(\cosh u - 1)$ which is stationary only at the point $u = 0$. Immediately, therefore, we can write down the approximation [15]

$$I \approx \left(\frac{2\pi}{kr_0'}\right)^{1/2} e^{i\pi/4} \int_0^{\pi} F(0, v)\ dv.$$

This approximation is good, of course, only if $\int_0^{\pi} F(u, v)\ dv$ is a slowly varying function of u. But this will always be the case if S is smooth enough. If, on the other hand, there are projections on S which are away from the line OP but which nevertheless contain points of reflection, then surely these must also be taken into account in evaluating the integral. In what follows we shall assume that no such projections exist.

The line $u = 0$, $0 \leq v \leq \pi$, to which we have reduced the surface integral, is the line segment $O'P'$. Adopting at this point a more suitable one dimensional notation, let us denote the distance r_0' by the letter x and the distance of the point $(0, v)$ from O by s. Then,

$$s = \tfrac{1}{2}x(1 - \cos v),$$

so that our approximation becomes

$$\left(\frac{2\pi}{kx}\right)^{1/2} e^{i\pi/4} \int_0^{x} F(s) \frac{ds}{[s(x - s)]^{1/2}}.$$

The final form of our integral equation can be further simplified by making several more approximations in $F(s)$. Thus we assume that da/da' is close to unity, that $1/kr_2$ $\partial r_2/\partial n$ is negligible, and that whenever r_0 , r_1 , r_2 appear in the modulus of $F(s)$ they can be replaced by the horizontal distances x, s, $x - s$, respectively. Equation (10) then becomes

$$W(x) = g(x) - e^{-i\pi/4}\left(\frac{k}{2\pi}\right)^{1/2} \int_0^{x} W(s)\left(\delta + \frac{\partial r_2}{\partial n}\right)e^{ik(r_1+r_2-r_0)}\left[\frac{x}{s(x - s)}\right]^{1/2} ds. \qquad (11)$$

This is the integral equation we are proposing. It is, so to speak, an approximate representation of our problem which has the advantage of simplicity while still retaining most of the characteristics upon which one intuitively feels the problem should depend. Actually, the reduction to one dimension is not an unfamiliar concept. Experimental and theoretical investigators have for a long time drawn profiles of the earth and attempted to correlate the profile with measured field strengths. Following our own

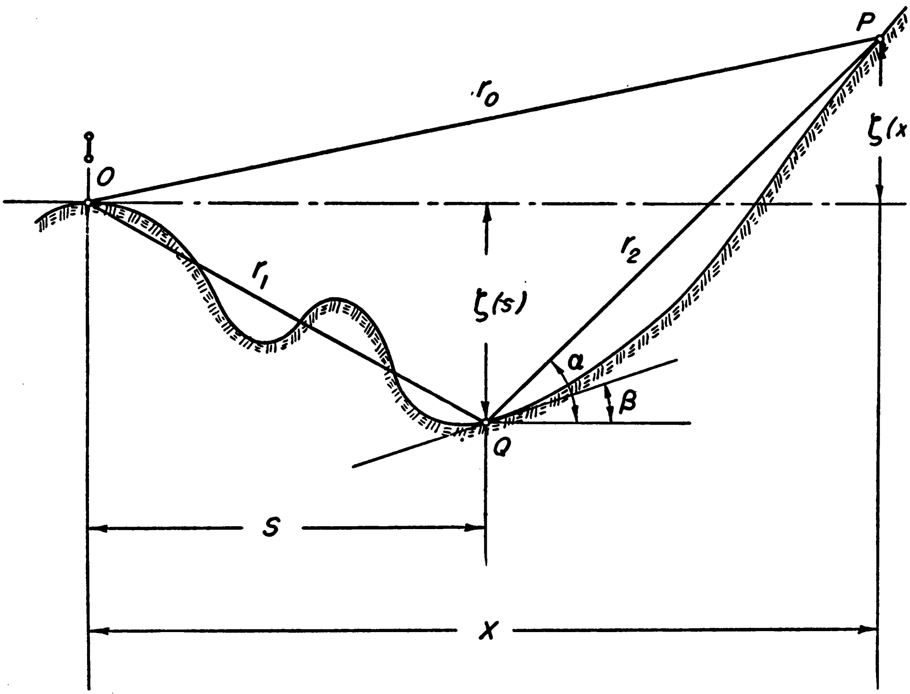

FIG. 2.

proposal, we would draw a profile similar to the one in Fig. 2; compute from it the quantities r_0, r_1, r_2, $\partial r_2/\partial n$; substitute these in Eq. (11); and then, by using suitable numerical methods, solve for $W(x)$.

Indeed, let us represent the profile in question by the function $\zeta(x)$. This function will be the elevation of the earth above a datum plane passing through the point O. Then we can write

$$r_0 = [x^2 + \zeta^2(x)]^{1/2} \approx x + \frac{\zeta^2(x)}{2x}$$

and similar expressions for r_1 and r_2. A little algebraic manipulation of these expressions gives us

$$r_1 + r_2 - r_0 \approx \frac{sx}{2(x - s)} \left[\frac{\zeta(s)}{s} - \frac{\zeta(x)}{x} \right]^2. \tag{12}$$

Similarly,

$$\frac{\partial r_2}{\partial n} = \sin (\beta - \alpha) \approx \zeta'(s) - \frac{\zeta(x) - \zeta(s)}{x - s}. \tag{13}$$

The field above the earth. We have thus found a procedure for evaluating $W(x)$ and hence the field ψ at all points of the earth's surface. It remains for us to show how this can be used to find the field at points in the space above this surface.

Using Green's theorem again together with Eq. (1) we find

$$4\pi \int_V \tau(Q) \frac{e^{ikR_2}}{R_2} \, dv = 4\pi\psi(P) - \int_S \left[\psi(Q) \frac{\partial}{\partial n} \frac{e^{ikR_2}}{R_2} - \frac{e^{ikR_2}}{R_2} \frac{\partial \psi(Q)}{\partial n} \right] da,$$

where this time P is in the space above S. The quantity R_2 is the radial distance PQ; we use the capital letter to stress the fact that P has been raised off the earth. Note that to exclude P from the integral of Green's theorem we now must construct a complete sphere about P rather than a hemisphere. It is for this reason that the $2\pi\psi(P)$ has changed to $4\pi\psi(P)$.

Now this equation, too, contains a surface integral whose integrand oscillates rapidly everywhere except possibly between O and P. Hence we can follow exactly the same steps we used before to change the integral to an approximate one-dimensional integral. If we replace the volume integral by $\psi_0(P)$ and use Eq. (2) to dispose of $\partial\psi/\partial n$, we shall finally arrive at the equation

$$\psi(P) = \psi_0(P) - e^{-i\pi/4}\left(\frac{k}{2\pi}\right)^{1/2} \int_0^x W(s)\left(\delta + \frac{\partial R_2}{\partial n}\right) e^{ik(r_1+R_2)} \left[\frac{x}{s(x-s)}\right]^{1/2} ds. \tag{14}$$

The quantities $r_1 + R_2$ and $\partial R_2/\partial n$ can be approximated by suitable modifications of Eqs. (12) and (13).

A plane earth. Although Eqs. (11) and (14) were derived largely to provide numerical solutions, it is perhaps of some interest to examine a few special cases which can be solved analytically.

One such case is that of a plane, homogeneous earth with an isotropic antenna at a height h_1 above the point O. Then $\partial r_2/\partial n = 0$, $r_1 + r_2 - r_0 = 0$, and

$$\psi_0(x) = (x^2 + h_1^2)^{-1/2} \exp\,[ik(x^2 + h_1^2)^{1/2}] \approx \frac{1}{x} \exp\left[ik\left(x + \frac{h_1^2}{2x}\right)\right]$$

or

$$g(x) \approx \exp\,(ikh_1^2/2x). \tag{15}$$

Now it may be argued that this last approximation can hardly be valid since at the point O where $x = 0$, the error becomes infinite. But while this will certainly affect the solution near O it will actually have very little affect further away. This is because the phase of the error also becomes infinite causing the integrated error to be small.

Introducing, then, these quantities into Eq. (11) we have

$$W(x) = \exp\,(ikh_1^2/2x) - e^{-i\pi/4}\delta\left(\frac{k}{2\pi}\right)^{1/2} \int_0^x W(s)\left[\frac{x}{s(x-s)}\right]^{1/2} ds. \tag{16}$$

This we can simplify further with the use of Sommerfeld's complex numerical distance. We define

$$\rho = i\frac{k\delta^2}{2}x \qquad \nu = i\frac{k\delta^2}{2}s,$$

whereupon Eq. (16) becomes

$$W(\rho) = e^{-a^2/4\rho} + i\pi^{-1/2} \int_0^\rho W(\nu)\left[\frac{\rho}{\nu(\rho - \nu)}\right]^{1/2} d\nu, \tag{17}$$

where $a = kh_1\delta$.

This integral equation can now be solved in any of several ways—by using the Liouville-Neumann series or by a method similar to that used in solving Abel's integral equation. We shall find it most convenient to use Laplace transforms. This we do first by defining

$$U(\rho) = \rho^{-1/2}W(\rho); \tag{18}$$

then Eq. (17) becomes

$$U(\rho) = \rho^{-1/2}e^{-a^2/4} + i\pi^{-1/2} \int_0^\rho \frac{U(\nu)}{(\rho - \nu)^{1/2}}\, d\nu. \tag{19}$$

In this equation the integral is of the *Faltung* or convolution type, so that when we apply the Laplace transformation to both sides we obtain [16]

$$L\{U\} = \left(\frac{\pi}{p}\right)^{1/2} e^{-ap^{1/2}} + ip^{-1/2}L\{U\}$$

or, solving for $L\{U\}$,

$$L\{U\} = \frac{\pi^{1/2}e^{-ap^{1/2}}}{p^{1/2} - i} = \left(\frac{\pi}{p}\right)^{1/2} e^{-ap^{1/2}} + i\left(\frac{\pi}{p}\right)^{1/2} \frac{e^{-ap^{1/2}}}{p^{1/2} - i}. \tag{20}$$

It only remains for us to find the inverse transform of this function. But the first term is the transform of $\rho^{-1/2}e^{-a^2/4}$, while the second term we can write as $i(\pi/p)^{1/2}f(p^{1/2})$ which is the Laplace transform of [17]

$$i\rho^{-1/2} \int_0^\infty e^{-t^2/4\rho}V(t)\, dt,$$

where

$$V(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{e^{-ap}}{p - i}\right\} = \begin{cases} 0 & t < a, \\ e^{i(t-a)} & t \geq a. \end{cases}$$

(This last, of course, assumes that a is real. But the functions that we obtain in the end will all be analytic in a and hence valid also for complex values.) Thus

$$\begin{aligned} U(\rho) &= \rho^{-1/2}e^{-a^2/4} + i\rho^{-1/2} \int_a^\infty \exp\left[-\frac{t^2}{4a} + i(t - a)\right] dt \\ &= \rho^{-1/2}e^{-a^2/4} + i\pi^{1/2}e^{-\rho - ia} \operatorname{erfc}\left(-i\rho^{1/2} + \frac{a}{2\rho^{1/2}}\right), \end{aligned} \tag{21}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty e^{-u^2}\, du.$$

Or, if we define

$$w = \rho\left(1 + \frac{ia}{\rho}\right)^2 = \rho\left(1 + \frac{h_1}{\delta x}\right)^2, \tag{22}$$

we have finally

$$W(\rho) = \rho^{1/2}U(\rho) = \exp\,[ikh_1^2/2x][1 + i(\pi\rho)^{1/2}e^{-w}\,\mathrm{erfc}\,(-iw^{1/2})]. \tag{23}$$

To complete our analysis, suppose that we raise the receiving antenna to a height h_2 . From Eq. (14) together with the usual approximations for R_2 and $\partial R_2/\partial n$, we have

$$\psi(P) = \frac{e^{ikR_0}}{R_0} + i\left(\frac{\rho}{\pi}\right)^{1/2} e^{ikx} \int_0^\rho \frac{W(\nu)}{\nu^{1/2}}\left[1 - \frac{ikh_2}{2(\rho - \nu)}\right](\rho - \nu)^{-1/2} \exp\,[-k^2h_2^2\delta^2/4(\rho - \nu)]\,d\nu, \tag{24}$$

where R_0 is the radial distance from the isotropic antenna to the point P.

In order to evaluate this integral we shall again make use of the *Faltung* theorem. Denoting the integral by I and remembering Eq. (20),

$$\begin{aligned} L\{I\} &= L\{\rho^{-1/2}W(\rho)\}L\left\{\left(1 - \frac{ikh_2}{2\rho}\right)\rho^{-1/2}\exp\,(-k^2h_2^2\delta^2/4\rho)\right\} \\ &= i\pi \exp\,[-k(h_1 + h_2)\,\delta p^{1/2}]\left[\frac{1}{p^{1/2}} - \frac{2}{p^{1/2} - i}\right]. \end{aligned} \tag{25}$$

Thus the Laplace transform of I is represented as the sum of two terms, the first of which is the transform of $i(\pi/\rho)^{1/2}\exp\,[-k^2(h_1 + h_2)^2\delta^2/4\rho]$, while the second is exactly the same transform as that in Eq. (20) except for the factor $-i2\pi^{1/2}$ and except that h_1 has been replaced by $h_1 + h_2$. Therefore we have immediately

$$\psi(P) = \frac{e^{ikR_0}}{R_0} - \frac{1}{x}\exp\,\{ik[x + (h_1 + h_2)^2/2x]\}\{1 - 2[1 + i(\pi\rho)^{1/2}e^{-w}\,\mathrm{erfc}\,(-iw)^{1/2}]\}, \tag{26}$$

where now

$$w = \rho\left(1 + \frac{h_1 + h_2}{\delta x}\right)^2. \tag{27}$$

The problem of electromagnetic radiation from a vertical Hertzian dipole over a plane and homogeneous earth was first successfully attacked by Sommerfeld in 1909, and since then many authors have treated its various aspects. To compare our Eq. (26) with the classical solutions, we quote here an approximation due to Norton: [18]

$$\psi(P) = \frac{e^{ikR_0}}{R_0} - \frac{e^{ikR_0'}}{R_0'}\{1 - 2[1 + i(\pi\rho)^{1/2}e^{-w}\,\mathrm{erfc}\,(-iw)^{1/2}]\},$$

where $\psi(P)$ is the Hertz potential at P,

$$\rho = ikR_0'/2n^2,$$

$$w = \rho[1 + n(h_1 + h_2)/R_0']^2,$$

n is the coefficient of refraction that we mentioned in Eq. (3), and R_0' is the radial distance from the dipole to the image of the point P,

$$R_0'^2 = x^2 + (h_1 + h_2)^2.$$

The agreement, it will be noticed, is almost exact.

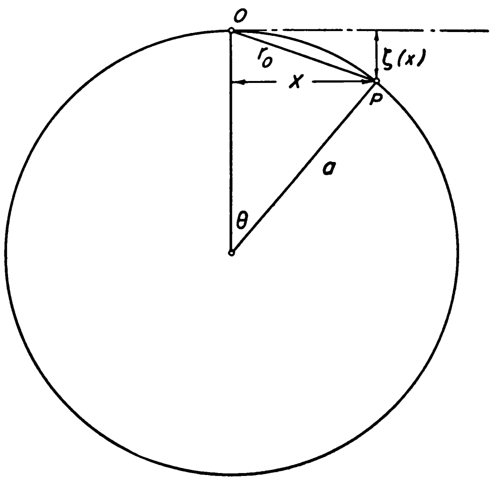

FIG. 3.

A spherical earth. As a last example let us consider an isotropic antenna at the surface of a spherical, homogeneous earth. If the sphere is of radius a, then from Fig. 3 we have

$$\zeta(x) = (a^2 - x^2)^{1/2} - a \approx -\frac{x^2}{2a}. \tag{28}$$

Substituting this approximation into Eqs. (12) and (13) gives us

$$r_1 + r_2 - r_0 = \frac{sx(x - s)}{8a^2},$$

$$\frac{\partial r_2}{\partial n} = \frac{x - s}{2a}. \tag{29}$$

Now we introduce the so-called natural units ξ, ζ, and γ, which are defined by

$$\xi = xa^{-2/3}(k/2\pi)^{1/3},$$

$$\zeta = sa^{-2/3}(k/2\pi)^{1/3}, \tag{30}$$

$$\gamma = i/(ka)^{1/3},$$

and further make the substitution

$$W(\zeta) = \xi^{-1/2}e^{i(\pi/12)\xi^3}U(\xi). \tag{31}$$

Then our problem of a spherical earth becomes immediately one of solving the integral equation

$$\begin{aligned} U(\xi) = \xi^{-1/2} \exp\left[-i\left(\frac{\pi}{12}\right)\xi^3\right] - \\ - \int_0^\xi U(\zeta) \exp\left[-i\left(\frac{\pi}{12}\right)(\xi - \zeta)^3\right]\left[\frac{e^{i\pi/4}}{\gamma(2\pi)^{1/3}} + \frac{1}{2}e^{-i\pi/4}(\xi - \zeta)\right](\xi - \zeta)^{-1/2}\, d\zeta. \end{aligned} \tag{32}$$

Again we have a *Faltung* integral. If we denote by $u(p)$ the Laplace transform of $U(\xi)$ and by $v(p)$ the transform of the function $\xi^{-1/2}e^{-i(\pi/12)\xi^3}$, then taking the Laplace transformation of both sides of Eq. (32) and making use of the *Faltung* theorem and the rule for differentiating a transform, we find easily

$$u(p) = \frac{v(p)}{1 + e^{i\pi/4}(2\pi)^{-1/3}\gamma^{-1}v(p) - (1/2)e^{-i\pi/4}v'(p)}. \tag{33}$$

And now to find $U(\xi)$ we need only to use the complex inversion formula

$$U(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u(p)e^{p\xi}\, dp. \tag{34}$$

Leaving aside, for the moment, all questions of existence and convergence we remark that this integral can be evaluated from a knowledge of the poles of $u(p)$. As we shall show later, $v(p)$ is an integral function so that the poles of $u(p)$ are all at the zeros of the denominator in Eq. (33). It follows that if these zeros are at the points $p = p_n$, $n = 0, 1, \cdots$, we shall finally have

$$U(\xi) = \sum_{n=0}^{\infty} \frac{v(p_n)e^{p_n\xi}}{e^{i\pi/4}(2\pi)^{-1/3}\gamma^{-1}v'(p_n) - (1/2)e^{-i\pi/4}v''(p_n)}. \tag{35}$$

The properties of $v(p)$. We have defined $v(p)$ by the integral

$$v(p) = \int_0^\infty \xi^{-1/2}e^{i(\pi/12)\xi^3}e^{-p\xi}\, d\xi \tag{36}$$

which, however, converges only for Re $p \geq 0$. If we make the transformation

$$\xi = (12/\pi)^{1/3}e^{-i\pi/6}t, \tag{37}$$

then it follows that

$$v(p) = (12/\pi)^{1/6}e^{-i\pi/12}k[(12/\pi)^{1/3}e^{-i\pi/6}p], \tag{38}$$

where

$$k(z) = \int_0^\infty t^{-1/2}e^{-t^3-zt}\, dt. \tag{39}$$

Since this last integral is analytic for all z it defines the analytical continuation of $v(p)$ to the entire p-plane.

Obviously $k(z)$ is regular everywhere and is therefore an integral function. It has the Taylor series expansion

$$k(z) = \int_0^\infty t^{-1/2}e^{-t^3} \sum_{n=0}^\infty \frac{(-zt)^n}{n!}\, dt = \frac{1}{3} \sum_{n=0}^\infty \frac{\Gamma(n/3 + 1/6)}{\Gamma(n + 1)} (-z)^n. \tag{40}$$

Now from this Taylor series, it is possible to derive an asymptotic expansion valid for large z. We need to know that the function $\Gamma[(z/3) + (1/6)]/\Gamma(z + 1)$ has poles at $z = -\frac{1}{2}, -3\frac{1}{2}, \cdots$, and that everywhere to the right of the imaginary axis it has the asymptotic expansion

$$\frac{\Gamma(z/3 + 1/6)}{\Gamma(z + 1)} \sim \pi^{1/2} \frac{2^{4/3}}{3^{1/2}} \left(\frac{2^{2/3}}{3}\right)^z \frac{1}{\Gamma(2z/3 + 4/3)} \left[1 + 0\left(\frac{1}{z}\right)\right].$$

Then it can be shown that [19]

$$k(z) \sim \begin{cases} \pi^{1/2}z^{-1/2} & -\dfrac{2\pi}{3} < \arg z < \dfrac{2\pi}{3}, \\ \pi^{1/2}[iz^{-1/2} \exp \{i(2/3^{3/2})z^{3/2}\} + z^{-1/2}] & \dfrac{2\pi}{3} \le \arg z \le \pi, \\ \pi^{1/2}[iz^{-1/2} \exp \{i(2/3^{3/2})z^{3/2}\} - z^{-1/2}] & \pi \le \arg z \le \dfrac{4\pi}{3}. \end{cases} \tag{41}$$

From this it follows that the zeros of $k(z)$ lie on or near the radials $e^{i2\pi/3}$ and $e^{i4\pi/3}$ since it is there that the exponentials of Eq. (41) have an absolute value equal to one. If we return to a consideration of the original function $v(p)$ we see from Eq. (38) that its zeros will lie along the radials $e^{i3\pi/2}$ and $e^{i5\pi/6}$. In fact if we let

$$p = \sigma e^{i3\pi/2} = \tau e^{i5\pi/6} \tag{42}$$

then for small arguments of σ and τ we have respectively

$$\begin{aligned} v(p) &\sim \pi^{1/2}e^{-i\pi/4}\sigma^{-1/2}\{\exp [i(4/3\pi^{1/2})\sigma^{3/2}] - \exp (-i\pi/2)\}, \\ v(p) &\sim \pi^{1/2}e^{i\pi/12}\tau^{-1/2}\{\exp [-i(4/3\pi^{1/2})\tau^{3/2}] - \exp (i\pi/2)\}, \end{aligned} \tag{43}$$

and further

$$\begin{aligned} v'(p) &\sim -2e^{-i\pi/4} \exp [i(4/3\pi^{1/2})\sigma^{3/2}], \\ v'(p) &\sim -2e^{-i\pi/4} \exp [-i(4/3\pi^{1/2})\tau^{3/2}], \\ v''(p) &\sim -4\pi^{-1/2}e^{i5\pi/12}\tau^{1/2} \exp [-i(4/3\pi^{1/2})\tau^{3/2}]. \end{aligned} \tag{44}$$

We shall need one more property of the function $v(p)$. It can easily be shown, after integrating Eq. (39) by parts, that

$$6k''' + 2zk' + k = 0. \tag{45}$$

On the other hand

$$\frac{d}{dz}\left[kk'' - \frac{1}{2}(k')^2 + \frac{z}{6}k^2\right] = \frac{1}{6}k[6k''' + 2zk' + k] = 0,$$

so that

$$kk'' - \frac{1}{2}(k')^2 + \frac{z}{6}k^2 = \text{const.} \tag{46}$$

The constant of integration, which can be determined by evaluating Eq. (40) and its derivatives at $z = 0$, is equal to $\pi/6$.

From this we can conclude that whenever $k(z) = 0$, then $k'(z) = \pm i(\pi/3)^{1/2}$. Or again, whenever $v(p) = 0$, then

$$v'(p) = \pm 2e^{i\pi/4}. \tag{47}$$

Furthermore, it follows from Eqs. (43) and (44) that at those zeros lying along the radial $e^{i3\pi/2}$ it is the positive sign that is valid, while along the radial $e^{i5\pi/6}$ it is the negative sign.

The field. Having thus determined the salient properties of the function $v(p)$ we are now in a position to discuss the properties of $u(p)$ and its inverse transformation.

First of all, it can be shown from Eq. (33) and from the asymptotic expansions of $v(p)$ that, except in the neighborhood of its poles, $u(p) = 0(p^{-1/2})$. From this it follows [20] that the integral of Eq. (34) converges, that it represents the function $U(\xi)$, and that it can be expanded in the series of Eq. (35). Thus this series is established rigorously as the solution to Eq. (32).

Now ordinarily γ is a very small quantity so that we shall expect the zeros of the denominator in Eq. (33) to be very near the zeros of $v(p)$. Thus there will be two sets of zeros: those in the vicinity of the radial $e^{i3\pi/2}$ and those near the radial $e^{i5\pi/6}$. And as a matter of fact, it follows from Eq. (47) that those zeros of $v(p)$ lying near the radial $e^{i3\pi/2}$ are exactly zeros of the denominator. But because they are zeros of $v(p)$ they can contribute nothing to the series of Eq. (35) since the numerator of each term contains a factor $v(p_n)$.

Thus we are left with only those zeros that lie near the radial $e^{i5\pi/6}$. If we write $p_n = \tau_n e^{i5\pi/6}$ and make use of the asymptotic approximations in Eqs. (43) and (44), then finding the zeros becomes equivalent to solving the equation

$$e^{-i(4/3\pi^{1/2})\tau_n^{3/2}} = -i\,\frac{1 + e^{-i\pi/6}(\pi/4)^{1/6}/\gamma\tau_n^{1/2}}{1 - e^{-i\pi/6}(\pi/4)^{1/6}/\gamma\tau_n^{1/2}} \tag{48}$$

or alternatively

$$\tan\left(\frac{\pi}{4} - \frac{2\tau_n^{3/2}}{3\pi^{1/2}}\right) = -e^{i\pi/3}\left(\frac{\pi}{4}\right)^{1/6}\frac{1}{\gamma\tau_n^{1/2}}. \tag{49}$$

And now we can write down the final answer. But before we do let us note that because of the geometry of the sphere

$$\begin{aligned} kr_0 + \frac{\pi}{12}\xi^3 &= k\left(r_0 + \frac{x^3}{24a^2}\right) \\ &= ka\left(\theta - \frac{\theta^3}{24} + \cdots + \frac{\theta^3}{24} + \cdots\right) \qquad (50) \\ &\approx ka\theta. \end{aligned}$$

Then it immediately follows from Eqs. (9), (31), (35), (48), and (50) that

$$\psi(P) = \frac{e^{ika\theta}}{r_0} 2e^{i\pi/4}\xi^{1/4} \sum_{n=0}^{\infty} \frac{\exp\left(-\tau_n e^{-i\pi/6}\xi\right)}{e^{i\pi/3}\pi^{-1}\tau_n - (2\pi)^{-2/3}\gamma^{-2}}. \tag{51}$$

Once again we have a solution which we can compare with the classical solutions for electromagnetic radiation; and happily enough we shall find much that is similar. Indeed, van der Pol and Bremmer [21] in finding the Hertz potential excited by a vertical electric dipole at the surface of a spherical, homogeneous earth, have suggested an equation which is—even to the extent of defining δ as in Eq. (3)—identical with Eq. (51). Furthermore, Eq. (49) is identical with their so-called tangent approximation to define the τ_n. It is of some interest to note that Fock [22] in still another attack on this same problem agrees even more directly with our results. He derives a contour integral which is very similar in form to Eq. (34) and then deduces a series expansion which is again identical with Eq. (51).

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