$$
\begin{aligned}
C_D &= 2h(1+Q)^{1/2}\left[1-\cos\beta+\frac{Q}{\pi}\sin\beta\log\sec\beta\right],\\
C_1 &= \frac{2h}{(1+Q)^{1/2}}\left[1-\cos\beta+\frac{Q}{\pi}\sin\beta\log\sec\beta\right].
\end{aligned}
\tag{36}
$$

The foregoing general solution for $Q$ small is bound by the condition that $\beta$ should not be small in comparison with $Q$. This merely implies an upper limit to the permissible width of jet and is no handicap in practice. Within the practical range of blockage ratios and cavitation numbers, the solution holds good.

When $Q$ is very small, the following first approximations may be used:

$$
\begin{aligned}
\frac{1}{h} &= S_5(\beta),\\
1 &= \frac{2h}{Q}\sin\beta,\\
a &= \frac{2h}{Q}(1-\cos\beta),\\
C_D &= C_1 = 2h(1-\cos\beta).
\end{aligned}
\tag{37}
$$

When $Q \to 0$, those results become those for the infinite cavity discussed in Part I.

It is not part of the present object to give detailed numerical results for application to arbitrary configurations: these it is hoped to present elsewhere.

Acknowledgement is made to the Chief Scientist, British Ministry of Supply, for permission to publish Part II of this paper. The views expressed in the paper are those of the authors.

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# A NEW VARIATIONAL PRINCIPLE FOR ISENERGETIC FLOWS*

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In a paper by Rubinov and the present author,[1] it is shown that the variational principle for irrotational flows of a compressible gas can be generalized to isenergetic flows. The functions to be varied are the stream function and the density distribution.

*Received Nov. 8, 1950.

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[1]Lin, C. C. and Rubinov, S. I. *On the flow of curved shocks,* J. Math. and Phys. **27**, 105-129 (1948).

For such flows, L. Crocco has introduced a new stream function, which depends on the entropy. The advantage of this apparent complication is to make the velocity components directly expressible in terms of the partial derivatives of this new function. Thus a single differential equation containing only this stream function can be more explicitly obtained. In this paper, it will be shown that the same integral used in earlier variational principles yields Crocco's equation when his stream function is being varied.

Following Crocco, we shall refer all speeds to the maximum speed attainable in the field. The pressure $p$ and the density $\rho$ are referred to suitable units consistent with this choice of typical speed. Then the isentropic acoustic speed $c$ is

$$c^2 = \gamma \frac{p}{\rho} \tag{1}$$

for an ideal gas with ratio of specific heats $\gamma$. The condition of constant energy may be rewritten as

$$c^2 = \frac{\gamma - 1}{2} (1 - w^2), \tag{2}$$

where $w$ is the total speed. If the stagnation density for the stream-line of zero (reference) entropy is taken as reference, the density is given by

$$\rho e^{S/R} = (1 - w^2)^{1/(\gamma - 1)}, \tag{3}$$

where $S$ is the entropy and $R$ is the universal gas constant.

Crocco's stream function $\Psi$ is so defined that the velocity components $u$ and $v$ along the directions of increasing $x$ and $y$ are given by

$$\begin{aligned} y^e u(1 - w^2)^{1/(\gamma-1)} &= \Psi_y , \\ y^e v(1 - w^2)^{1/(\gamma-1)} &= -\Psi_x , \end{aligned} \tag{4}$$

where $e = 0$ is the two-dimensional case and $e = 1$ is the axially symmetrical case. In the latter case, as usual, $x$ is taken along the axis of symmetry and $y$ is in a perpendicular direction. The vorticity is

$$\omega = v_x - u_y = y^e(1 - w^2)^{\gamma/(\gamma-1)} g(\Psi), \qquad g(\Psi) = \frac{\gamma - 1}{2\gamma R} S'(\Psi).$$

When $u$ and $v$ are substituted from (4), this leads to Crocco's equation for the stream function $\Psi$:

$$\left(1 - \frac{u^2}{c^2}\right)\Psi_{xx} - \frac{2uv}{c^2}\Psi_{xy} + \left(1 - \frac{v^2}{c^2}\right)\Psi_{yy} - e\frac{\Psi_y}{y} = y^{2e}(1 - w^2)^{(\gamma+1)/(\gamma-1)}\left(\frac{w^2}{c^2} - 1\right). \tag{5}$$

We shall now show that

$$\delta I = \delta \iint (p + \rho w^2) y^e \, dx \, dy = 0 \tag{6}$$

with suitable boundary conditions, will lead to Eq. (5). By use of (1) and (2), the integral $I$ becomes

$$I = \frac{1}{2\gamma} \iint A(w^2) \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] y^{\epsilon}\, dx\, dy \tag{7}$$

with

$$A(w^2) = \{(\gamma - 1) + (\gamma + 1)w^2\}(1 - w^2)^{1/(\gamma-1)},$$

where $\Psi_0$ corresponds to the streamline along which $S = 0$. In this integral, $w$ is supposed to be expressed in terms of $\Psi_x$ and $\Psi_y$ through (4). In particular, $w^2$ can be expressed in terms of $\Psi_x^2 + \Psi_y^2$ by

$$y^{2\epsilon}B(w^2) = \Psi_x^2 + \Psi_y^2, \qquad B(w^2) = w^2(1 - w^2)^{2/(\gamma-1)} \tag{8}$$

The variation $\delta I$ consists of two parts: (1) the direct variation of $\Psi$, (2) the variation of $\Psi$ through $w^2$. The latter part can be easily transformed into variation of $\Psi$ by (8). By noting that

$$A'(w^2)/B'(w^2) = \gamma(1 - w^2)^{-1/(\gamma-1)}$$

we obtain

$$\delta I = \frac{-1}{\gamma - 1} \iint A(w^2) \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] g(\Psi)\, \delta\Psi\, y^{\epsilon}\, dx\, dy$$

$$+ \iint (1 - w^2)^{-1/(\gamma-1)}\{\Psi_x(\delta\Psi)_x + \Psi_y(\delta\Psi)_y\} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] y^{-\epsilon}\, dx\, dy.$$

With boundary condition $\Psi_n \delta\Psi = 0$, $\delta I = 0$ leads to the equation

$$-\frac{\partial}{\partial x}\left\{(1 - w^2)^{-1/(\gamma-1)} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] \frac{\Psi_x}{y^{\epsilon}}\right\}$$

$$+ \frac{\partial}{\partial y}\left\{(1 - w^2)^{-1/(\gamma-1)} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] \frac{\Psi_y}{y^{\epsilon}}\right\}$$

$$= -\frac{A(w^2)}{\gamma - 1} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] g(\Psi) y^{\epsilon}.$$

By direct differentiation and by use of (4) and (2), Eq. (5) is verified.

For convenience of comparison, we record that

$$\frac{d\Psi}{d\psi} = e^{S/R}$$

in the present dimensionless form, where $\psi$ is the usual stream function.