QUARTERLY

OF

APPLIED MATHEMATICS

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5. Conclusions

By using an expansion of the type (2.1) of the determinantal frequency equation (1.3), obtained for ordinary mechanical systems with n degrees of freedom, it is easy to obtain approximate expressions (3.1), supposing "weak coupling" and distinct eigenvalues. By examples it is shown, that similar developments of the eigenvalues as (3.1) will hold also in cases, where at least some of the terms in the principal diagonal of the matrix are of the same order as the coupling terms.

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BOOK REVIEWS

Electromagnetic problems of microwave theory. By H. Motz. Methuen & Co. Ltd., London, and John Wiley & Sons, Inc., New York, 1951. vii + 184 pp. $2.00.

This book is a new Methuen Monograph dealing in Chapter one with a summary of general topics in microwave methods including velocity modulated tubes, travelling wave tubes, resonators, cavity magnetrons, methods of detection, wavemeters, standing wave meters, the Smith chart, and matching. Chapters two and three deal with a detailed discussion of velocity modulation and Klystron theory following the work of Webster. Chapter four is entitled Mode Selection in Cavity Magnetrons and includes the Fourier Analysis of the rotating waves and conditions for synchronism of the electrons with the rotating field components. Some discussion of mode stability is given together with an estimate of magnetron efficiency.

Chapter five discusses the field relations in wave guides in orthogonal curvilinear systems. Application is made to guides of rectangular and circular cross section. The use of series expansions in terms of normal mode solutions is illustrated. The transmission line analogy is discussed briefly.

Calculation of electromagnetic fields in cavities and guides of complex shape is discussed in some detail in chapter six where relaxation and finite difference methods are used. The methods are illustrated with resonator gaps. The determination of the field pattern for higher modes is also discussed briefly. The analytical treatment of corrugated wave guides, as done by Wilkinshaw, is outlined very well in the space of about six pages.

Chapter seven is concerned with the impedance of an antenna in a wave guide. The problem of coupling into wave guides by means of straight wires or loops, as treated by the Toronto group, is presented in a very satisfactory way. The theory of discontinuities in wave guides as worked out by Bethe and Schwinger is outlined and applied to transverse windows, changes in cross section, and bends. The methods are applicable only to thin windows and sudden changes in cross section but a numerical method, applicable to any type of discontinuity, is outlined.

A large amount of useful material is packed into the 180 pages that make up this monograph. The writer did a very good job of discussing the problems which he includes. A person actively working in microwave theory will probably find this book useful as well as the person looking for general information.

Rohn Truell

Ordinary non-linear differential equations in engineering and physical sciences. By N. W. McLachlan. Oxford at the Clarendon Press, 1950. vi + 201 pp. $4.25.

The contents of this book are of essentially two categories. Following a general introduction, the second chapter deals with examples of non-linear equations which can be integrated explicitly. The third chapter is concerned with the particular situation where the explicit integration requires the use of elliptic functions. The remainder of the book deals with the second category, i.e., the non-linear equations which arise in vibration problems.

In chapter four, the Van der Pol equation with small parameter, and the non-linear restoring force problem, are treated in detail. Emphasis is given to the physical phenomena involved as well as to the mathematical solutions. The relaxation oscillations of the Van der Pol equation are discussed very briefly. The method of slowly varying amplitude and phase and the equivalent linear equation are discussed in chapters five and six. Chapter seven is devoted to equations with periodic coefficients and chapter eight to numerical and graphical methods of solutions. Almost every equation in the book is illustrated by a physical problem so that the book should be of considerable interest to engineers.

G. F. Carrier

Integraltafel. Zweiter Teil: Bestimmte Integrale. By Wolfgang Gröbner and Nikolaus Hofreiter. Springer-Verlag, Wien and Innsbruck, 1950. vi + 551 pp. $5.80.

This very complete table of definite integrals is a companion volume to Integraltafel (Unbestimmte Integrale) of indefinite integrals by the same authors, published in 1949 (reviewed QAM, January 1950). This book contains approximately two thousand definite integrals including some which are not found in Bierens de Haan. The symbols are large and clear and the general make-up of the table is very satisfactory. The only change that this reviewer would ask for is specification of the limits of integration in the table of contents. The table of contents is as follows: Symbols and Notation, Methods of Calculating Definite Integrals, General Integral Formulae. (1) Rational Integrands (detailed listing omitted). Orthogonal Polynomials, including Legendre, Hypergeometric, Tschebyscheff, Associated Legendre, Laguerre and Hermite Polynomials (detailed listing omitted). (2) Algebraic Irrational Integrands, including elliptic integrals of Legendre and Weierstrass Canonical forms (detailed listing omitted). (3) Elementary Transcendental Integrands. Integrands of the form $R(\exp \lambda x, \exp \mu x, \cdots)$, $[\exp(-sx)]f(x)$, $R(x, \exp \lambda x)$, $R(x, \exp f(x))$, $f(\log x)$, $\log [g(x)]$, Euler Dilogarithm, $f(x)(\log x)^n$ with $f(x)$ rational, irrational and transcendental, $f(x) \log [g(x)]$, $F[x, \log f(x)]$, (exponential, logarithmic, sine and cosine integrals), $f(\sin x, \cos x)$, $f(\sin ax, \cos bx, \cdots)$, $F[x, \sin ax, \cos bx]$, $F[x, \sin f(x), \cos g(x), \cdots]$, $F[\exp ax, \sin bx, \cos cx]$, $F[x, \exp ax, \sin bx, \cos cx]$, $F[x, \exp f(x), \sin g(x), \cos h(x)]$, $F[x, \log f(x), \sin g(x), \cos h(x)]$, $F[x, \text{arc sin } x, \text{arc cos } x]$, $F[x, \text{arc tan } x, \text{arc cotan } x]$, $R[\exp \lambda x, \sinh ax, \cosh bx]$, $R[x, \sinh ax, \cosh bx]$, $F[f(x), \sinh ax, \cosh bx]$. Integrals of $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\text{cotanh}^{-1} x$. (4) Euler Integrals. Gamma function, Beta function, products of powers of linear expressions with general exponents, products of powers of binomial expressions with general exponents, products of powers of higher order expressions with general exponents. (5) Integrals of Cylinder functions. Bessel functions, Bessel functions with imaginary arguments, Integrals of the form $F[x, Z_n(x)]$, $F[x, \exp x, \log x, Z_n(x)]$, $F[x, \sin x, \cos x, Z_n(x)]$, $F[x, Z_n(x), Z_\mu(x)]$.

Rohn Truell