THE ELASTIC AXES OF A ONE-MASS ELASTICALLY SUPPORTED SYSTEM*

BY J. J. SLADE, JR.

When an elastically supported rigid body is subjected to the action of a rectilinear sinusoidal force, the resulting steady motion generally consists of rectilinear and torsional oscillations with frequency equal to that of the exciting force. It is desired to determine the location of the exciting force so that the torsional oscillations are suppressed or, at least, so that the amplitude of these oscillations is reduced to a minimum. The problem arises, for example, in connection with unbalanced machines on elastic foundations, as well as in investigations of the dynamic characteristics of elastically supported rigid assemblies by means of induced vibrations.

The two-dimensional problem has been considered under simplifying conditions.[1] The three-mass mechanical oscillator[2] that produces a force the axis of which may be made to coincide with any line in a fixed plane, when the oscillator is in a fixed position, presents problems that require an extension of existing theory. The present investigation deals with the general case.

We consider a rigid body of mass m that can move freely under general linear elastic constraints with linear damping. Let r be the displacement of its center of gravity with respect to its position in static equilibrium. Since only small oscillations are considered, elastic and damping reactions may be taken to be fixed to a moving frame with origin at r.

Let $\Phi + \epsilon\Phi_0$ and $\Psi + \epsilon\Psi_0$ be dual dyadics such that $-(\Phi + \epsilon\Phi_0)\cdot r$ is the motor[3] of the elastic suspension and $-(\Psi + \epsilon\Psi_0)\cdot r'$ that of the damping system, due to a rectilinear displacement, the prime denoting differentiation with respect to time.

Finally let f be the exciter force and p a point on its line of action. It should be noted that in all cases considered the exciter is rigidly connected to, and forms part of the system. The exciter force is strictly fixed in the moving frame.

The motion of the center of gravity of the body is governed by the equation

$$mr'' + \Psi\cdot r' + \Phi\cdot r = f. \tag{1}$$

There is also the moment

$$c = p \times f - (\Phi_0\cdot r + \Psi_0\cdot r') \tag{2}$$

that tends to produce torsional oscillations.

If the angular frequency of the exciter force is ω, we may write

$$f, r, c = (F, R, C)e^{i\omega t}$$

where

$$(-m\omega^2 I + i\omega\Psi + \Phi)\cdot R = F \tag{3}$$

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[1]See, for example, E. Rausch, *Machinenfundamente und andere dynamische Bauaufgaben*, Ch. III, V.D.I., Berlin, 1936.

[2]R. K. Bernhard, *Study on mechanical oscillators*, Proc. Am. Soc. Test. Materials **29**, 1016-1036 (1949).

[3]L. Brand, *Vector and tensor analysis*, Ch. II, J. Wiley & Sons, New York, 1947.

and

$$C = p \times F - (\Phi_0 + i\omega\Psi_0)\cdot(-m\omega^2 I + i\omega\Psi + \Phi)^{-1}\cdot F$$
$$= p \times F + (\Gamma + i\Lambda)\cdot F. \tag{4}$$

Our problem is to determine under what conditions, if any, we can make $C = 0$ with $F \neq 0$.

An axis with which the line vector $F + \epsilon p \times F$ must coincide to satisfy these conditions fully is here called an elastic axis[4] of the system. An axis of fixed direction with which the line vector must coincide to make $|C| \neq 0$ a minimum will be called a quasi-elastic axis.

Suppose first that the system is conservative, so that $\Psi + \epsilon\Psi_0 = 0$. If oscillations about the axis of the free vector a are suppressed, then $a\cdot C = 0$; or, since in the conservative case $\Lambda = 0$,

$$a\cdot p \times F + a\cdot\Gamma\cdot F = 0. \tag{5}$$

Now, the left hand member of this equation is the moment of the fixed motor $a + \epsilon a\cdot\Gamma$ about the axis $F + \epsilon p \times F$ Whence:

Oscillations about an axis a are suppressed when the line vector $F + \epsilon p \times F$ coincides with a line of the null system of the motor $a + \epsilon a\cdot\Gamma$.

Let e_1 , e_2 , e_3 be unit vectors in the directions of the principal axes of the elastic suspension. In this presentation the diagonal elements of Γ are zero and $e_k\cdot e_k\cdot\Gamma = 0$, so that the motor $e_k + \epsilon e_k\cdot\Gamma$ is a line vector. We therefore have the following results.

1. Rotational oscillations about a principal axis are suppressed when the line of action of the exciter force coincides with a line of the special linear line complex the axis of which is $e_k + \epsilon e_k\cdot\Gamma$.

2. Rotational oscillations about two principal axes are simultaneously suppressed when the exciter force coincides with a line of the linear congruence, the directrices of which are $e_j + \epsilon e_j\cdot\Gamma$, $j = k, l$.

3. The elastic axes of the system are the lines of the regulus the directrices of which are $e_k + \epsilon e_k\cdot\Gamma$, $k = 1, 2, 3$.

When the system is not conservative, the following equation must be added to (5):

$$a\cdot\Lambda\cdot F = 0. \tag{6}$$

4. Oscillations about a are suppressed when the force coincides with a line of the null system of $a + \epsilon a\cdot\Gamma$ that is perpendicular to the fixed couple vector $a\cdot\Lambda$.

Assuming that the principal axes of Ψ coincide with those of Φ, as they generally do in practical cases, we may further state.

1. Rotational oscillations about a principal axis e_k are suppressed when $F + \epsilon p \times F$ is a line of the plane through $e_k + \epsilon e_k\cdot\Gamma$ perpendicular to $e_k\cdot\Lambda$.

2. When $F + \epsilon p \times F$ is the line of intersection of two such planes oscillations are simultaneously suppressed about the corresponding two principal axes.

In general the non-conservative system possesses no elastic axes. Since Γ and Λ are constants, when ω is fixed, we see from Equation 4) that, if F is held fixed, $|C|$ is a minimum when p is so determined that

$$p \times F + \Gamma\cdot F = 0. \tag{7}$$

[4]Rausch, *loc. cit.*, uses the terms *elastiche Hauptachse* and *elasticher Mittelpunkt*.

This leads to the following result: The quasi-elastic axes of the system are the lines of the regulus the directrices of which are $e_k + \epsilon e_k \cdot \Gamma$, $k = 1, 2, 3$.

The traces of the directrices $e_k + \epsilon e_k \cdot \Gamma$ on the principal planes have been called elastic centers. The locus of an elastic center starts, with $\omega = 0$, at a point that depends on the parameters of the elastic suspension and ends at the center of gravity ($\omega = \infty$).

In the conservative case this locus is the outside section of a hyperbola, the inside section corresponding to $\omega^2 < 0$. The locus is a 4th degree algebraic curve in the non-conservative case. The reduced system in which one reaction goes through the center of gravity and the other two lie in a plane through this center has been considered in detail.[5]

HEAVY DISK SUPPORTED BY CONCENTRATED FORCES*

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Muschelišvili solved the problem of a two-dimensional light disk subjected to an arbitrary number of concentrated forces by means of his method of complex variable [1, 273-274].** When the weight of the disk has to be taken into consideration, the problem may still be solved in a similar way. Muschelišvili's notations will be followed throughout this paper, and only additional ones will be defined as they first occur.

In plane problems including body forces due to gravity, the stress function U may still satisfy the biharmonic equation if it is defined by the following equations:

$$\frac{\partial^2 U}{\partial x^2} = \tau_{yy} - V_1, \qquad \frac{\partial^2 U}{\partial y^2} = \tau_{xx} - V_1, \qquad -\frac{\partial^2 U}{\partial x\, \partial y} = \tau_{xy} \tag{1}$$

in which V_1 is the body force potential due to gravity and is equal to wy when gravity acts in the negative y-direction [3], w being the specific weight of the material of the body. Hence, the function U may be expressed in terms of two analytic functions as shown by Muschelišvili [2, 284].

The boundary conditions

$$\tau_x = \tau_{xx}\frac{dy}{ds} - \tau_{xy}\frac{dx}{ds}, \qquad \tau_y = \tau_{xy}\frac{dy}{ds} - \tau_{yy}\frac{dx}{ds}$$

however, may be shown to lead to some different result. When stress components given by Eqs. (1) are substituted into these conditions and computations carried out in the same manner as given by Muschelišvili [2, 301-302], the following result is obtained:

$$\varphi_1(z) + z\bar{\varphi}_1'(\bar{z}) + \bar{\psi}_1(\bar{z}) = i\int (\tau_x + i\tau_y)\, ds - \int V_1\, dz$$

If we define

$$f_1 + if_2 = i\int (\tau_x + i\tau_y)\, ds - \int V_1\, dz \tag{2}$$

[5] R. K. Bernhard and J. J. Slade, Jr., *On the elastic center of one-mass plane oscillatory systems* (unpublished). Dynamics Laboratory, Bureau of Engineering Research, Rutgers University.

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**The first number in each square bracket refers to the References listed at the end of the paper. The subsequent numbers, if any, refer to the page numbers of the reference.