y_0 will be taken to be $A \sin t$. (This is exactly the problem solved by McLachlan in [1], p. 190). Then

$$p(t) = \left[\left\{\frac{A^2}{2} - 1\right\} - \frac{A^2}{2}\cos 2t\right],$$

$$q_1(t) = A^2 \sin 2t,$$

$$a_0 = \frac{A^2}{2} - 1.$$

$$a_2 = c_0 = d_2 = 0, \qquad c_2 = A^2; \qquad b_2 = -\frac{A^2}{2}$$

Notice that for periodicity we would need

$$\omega_1 = \frac{1}{2}\left(\frac{A^4}{16} - \frac{A^4}{4} + A^2 - 1\right)^{1/2} = 0,$$

since here $\omega = 1$. If $A^2 = 4$, then this condition is satisfied. Further, using (8) since $c_0 - 2\omega_1 = 0$ and $\frac{1}{4}\{(c_2 + b_2)^2 + (a_2 - d_2)^2\} > 0$

$$\lambda_1 = -\frac{A^2}{8} + \frac{1}{2}.$$

Thus $\lambda_1 < 0$ for $A^2 > 4$ and >0 for $A^2 < 4$.

This example as was remarked is given by McLachlan [1] and was presented merely to show the ease in which stability characteristics may be obtained once λ_1 is computed in terms of a_0 , a_2 , b_2 , c_2 , and d_2 .

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ON THE RELATIONSHIP BETWEEN THE MARTIENSSON AND DUFFING METHODS FOR NONLINEAR VIBRATIONS*

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The background for a number of one-term approximation methods and their application to forced nonlinear vibrations has recently been discussed by Schwesinger.[1]

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[1]G. Schwesinger, *On one-term approximations of forced nonharmonic vibrations*, J. Appl. Mech. **17**, 202-208 (1950). Note that he attributes to Rüdenberg the method that is designated here as Martiensson's method.

As he points out, such one-term approximations may be useful in problems of analysis when the response is close to sinusoidal, as is frequently the case for small nonlinearities. They are not the ultimate tool, however, for one can always go to approximations which contain the higher harmonic terms as well. The situation is different for problems of nonlinear synthesis. There, the nature of the problem is such that approximations of more than one term are virtually prohibited, and one is forced to accept answers that are given by the one-term methods. Thus, there is a body of problems for which such methods have intrinsic importance.

The requirements of synthesis have led to a re-examination of some of these one-term methods. For this purpose, it is important to know in a general way whether the use of the various methods leads to different synthesis. In particular, one should know which of the methods are really independent of one another, in order to know how many synthesis possibilities must be examined. It was found that the results obtained by the methods known by the names of Martiensson (or Rüdenberg) and Duffing[2] are not independent. For the class of equations to which both methods are applicable, there is a simple relationship between their results. The purpose of this note is to develop this relationship.

The fact that the results of the two methods are simply related is the more striking because their rationales are so different. Briefly, these are as follows. In the Martiensson method, since an assumed one-term solution does not generally satisfy the differential equation identically, one forces satisfaction at two points, say $t = 0$ and at the quarter-period. In this way, an algebraic equation is obtained for the amplitude of the one-term solution. To use the Duffing method, one writes the equation in a form for iteration, and puts an assumed one-term solution into this equation as a first approximation. In order to enforce the requirement that the next approximation be periodic, one is obliged to choose a certain coefficient equal to zero. This again leads to an algebraic equation for the amplitude. As a development of the Lindstedt perturbation method, the Duffing method is perhaps the more rational procedure of the two. (It is fundamentally different, of course, in that it permits extension to higher approximations.)

Now let us consider the methods in greater detail. They are usually illustrated only for systems with one degree of freedom, but it is not difficult to extend them to certain higher order systems. The Martiensson method, as ordinarily applied, is limited to systems without dissipation. The most general equation to which it seems appropriate, at least without major changes in formalism, is

$$\Omega^2 \ddot{x} + Lx + \nu f(x) = F \sin t. \tag{1}$$

Here, $x(t)$ and f are $(n \times 1)$ column matrices, F is a constant $(n \times 1)$ matrix, and L is a constant $(n \times n)$ matrix. Both Ω^2 and ν are scalar parameters, the former being related to the frequency of excitation and the latter being the nonlinearity parameter. Suppose that $f(x)$ is a continuous odd function, with $f(0) = 0$. We will seek a periodic solution with period 2π.

Let us apply the Martiensson method formally to Eq. (1). We assume an approximation solution $x^{(0)} = a \sin t$ (a being a column matrix of amplitudes), which satisfies

[2]We restrict our consideration to the first approximation by this method, in effect making it a one-term method.

the equation at time $t = 0$. If we require that it also satisfy the equation at $t = \pi/2$, we obtain

$$(\Omega^2 - L)a - \nu f(a) + F = 0. \tag{2}$$

On the other hand, let us write Eq. (1) in a form for iteration as

$$\Omega^2(\ddot{x}^{(1)} + x^{(1)}) = (\Omega^2 - L)x^{(0)} - \nu f(x^{(0)}) + F \sin t.$$

Using the same approximation as before,

$$\Omega^2(\ddot{x}^{(1)} + x^{(1)}) = [(\Omega^2 - L)a + F] \sin t - \nu f(a \sin t). \tag{3}$$

Since f is a periodic function of t with period 2π, we can write

$$f(a \sin t) = \mathfrak{F}_1(a) \sin t + \sum_{m=2}^{\infty} \mathfrak{F}_{2m-1} \sin (2m - 1)t.$$

Following the Duffing iteration method, we require that no secular term arise when Eq. (3) is solved for $x^{(1)}$. This means that we must put the coefficient of sin t in Eq. (3) equal to zero, namely

$$(\Omega^2 - L)a - \nu\mathfrak{F}_1(a) + F = 0. \tag{4}$$

Equation (4) is the analog of Eq. (2), and is identical except that the first Fourier coefficient of $f(a \sin t)$ replaces the value $f(a)$.

This result has limited usefulness in its general form. However, in systems which contain only one nonlinear element, f is particularly simple. It has the representation $f = M\varphi(x_i)$, where M is a column matrix and φ is a scalar function of only one of the x-components. As perhaps the simplest possible example, we may consider the case where $\varphi(x_i) = x_i^m$, i.e. a simple power function. Equations (2) and (4) become respectively

$$(\Omega^2 - L)a - \nu a_i^m + F = 0$$

$$(\Omega^2 - L)a - c\nu a_i^m + F = 0$$

where c is the constant whose value is the first Fourier sine coefficient of $\sin^m t$. For example, if $m = 3$, then $c = 3/4$.

The result of this simple special case has a very useful implication in synthesis problems. It means that if a synthesis is attempted for a system containing a single nonlinear element which obeys a power law, the system being described by Eq. (1), then the procedure follows identical paths for the Martiensson and Duffing methods. It is known at once that the optimum nonlinearity by one of the methods, say the Duffing, is just $1/c$ times as large as those predicted by the other, and that optimum values of any parameters contained in L are identical by the two methods. Nothing should be inferred, of course, as to which method is better as a one-term approximation, since this is an entirely separate problem.