# —NOTES—

## ON THE RATE OF CONVERGENCE OF RELAXATION METHODS*

By R. Plunkett (*The Rice Institute*)**

A recent paper by Frankel [1] gives the rates of convergence and a time estimate for the solution of finite approximations to Poisson's equations and the biharmonic equation by some of the standard iteration methods. This shows that in general the time required is prohibitive for a reasonably large number of points. It is well known that relaxation methods [2] are faster for hand computing but an estimate of rate of convergence is of interest for machine programming for the prospective use of digital calculators. In general the finite approximation to partial differential equations may be written:

$$Az + f = 0 \tag{1}$$

where $z$ is an $n$-element column matrix of the unknown values, and $n$ is the number of points taken in the region. $A$ is an $n \times n$ square matrix, with a high degree of regularity; the elements of the main diagonal are the largest and most of the other elements are zero; $f$ is a known column matrix. If $z_m$ is the $m$th approximation to $z$, the matrix of the residuals is defined by

$$Az_m + f = R_m \tag{2}$$

Then one element of $z_m$ is adjusted in such a way as to make the greatest reduction in the value of $R_m$. Thus, if $e_i$ is a unit vector of $A$-space, i.e., $e_i$ has zero for all its elements except the $i$-th element which is 1,

$$z_{m+1} = z_m - c_{m+1}e_i \tag{3}$$

So far the value of $R_m$ has not been defined, but it is clear that if all the elements of $R_m$ are reduced to zero $z_m = z$. The usual approach in the numerical solution by relaxation methods is intuitive, but it can easily be seen that if

$$c_{m+1} = \frac{e_i'R_m}{e_i'Ae_i} \tag{4}$$

then

$$e_i'R_{m+1} = 0.$$

Thus, this has the effect of reducing one element of $R_{m+1}$ to zero even though it may actually increase the others. It has been shown [3], if $A = A'$ and is a positive definite form, that

$$H_m = \tfrac{1}{2}z_m'Az_m + z_n'f \tag{5}$$

will be reduced by such a step and the process must ultimately converge. It is difficult to get an estimate for $H_m$, however, and so this is not the easiest criterion to use. In our case we shall use the more customary standard of

$$|\, R_m \,| = (R_m'R_m)^{1/2}. \tag{6}$$

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**Now with General Engineering Laboratory, General Electric Company, Schenectady, N. Y.

From Eqs. (3) and (2)

$$R_{m+1} = R_m - c_{m+1} A e_i \tag{7}$$

and

$$| R_{m+1} |^2 = | R_m |^2 - 2c_{m+1} R_m' A e_i + c_{m+1}^2 e_i' A' A e_i \tag{8}$$

with no restrictions on $A$. Thus

$$(| R_m | - | R_{m+1} |)(| R_m | + | R_{m+1} |) = 2c_{m+1} R_m' A e_i - c_{m+1}^2 e_i' A' A e_i \ . \tag{9}$$

Since in general $| R_m |$ and $| R_{m+1} |$ differ very little, letting $| R_m | - | R_{m+1} | = \Delta R_{m+1}$

$$\Delta R_{m+1} \approx \frac{2c_{m+1} R_m' A e_i - c_{m+1}^2 e_i' A' A e_i}{2 \, | R_m |}. \tag{10}$$

Let the elements of $R_m$ be random variables with a mean value of zero, a maximum value of $\rho_m$ and a standard deviation of $b\rho_m$ where $b$ is a number less than one. Then the mathematical expectation of $| R_m |$ is:

$$| R_m | = n^{1/2} b \rho_m \ . \tag{11}$$

The value of $c_{m+1}$ is found from Eq. (4) by taking $i$ such as to make $c_{m+1}$ take its largest value. From Eq. (10) the mathematical expectation of $\Delta R_{m+1}/| R_m |$ can be found.

From Eq. (4),

$$c_{m+1} = \frac{e_i' R_m}{e_i' A e_i} \approx \frac{\rho_m}{a_{ii}}$$

and

$$R_m' A e_i \approx a_{ii} \rho_m \ .$$

where $a_{ii}$ is a typical element of the main diagonal.

Likewise

$$e_i' A' A e_i = l a_{ii}^2 \ ,$$

where $l$ is a small number greater than one [for Poisson's equation $l = 1.25$, for the biharmonic $l = 1.72$].

Thus,

$$2c_{m+1} R_m' A e_i \approx 2\rho_m^2 \ ,$$

$$c_{m+1}^2 e_i' A' A e_i \approx l \rho_m^2$$

or, the mathematical expectation of

$$\frac{\Delta R_{m+1}}{| R_m |} = \frac{\rho_m^2 (2 - l)}{2nb^2 \rho_m^2} = \frac{1 - l/2}{b^2 n} = k \frac{1}{n} \ , \tag{12}$$

where $k$ is a number not much greater than one unless $b$, the standard deviation ratio, is very small. Thus for any reasonable distribution of the elements of $R_m$, the probability of $\Delta R_{m+1}/| R_m |$ being very much greater than $k/n$ is small, or the standard deviation of this quantity must be of the same order of magnitude as the quantity itself. Since because of the way it is found the correlation among these ratios for different $m$ is small, then

$$\frac{|R_{m+n}|}{|R_m|} = \prod_{i=1}^{n}\left(1 - \frac{\Delta R_{m+i+1}}{|R_{m+i}|}\right) \tag{13}$$

and

$$\log\frac{|R_{m+n}|}{|R_m|} = \sum_{i=1}^{n} \log\left(1 - \frac{\Delta R_{m+i+1}}{|R_{m+i}|}\right) = \sum_{i=1}^{n} - \frac{k}{n} - \frac{1}{2}\left(\frac{k}{n}\right)^2 - \cdots$$

Thus the mathematical expectation of

$$\log\frac{|R_{m+n}|}{|R_m|} = -k + o\left(\frac{1}{n}\right)$$

and a good working value for

$$r_n = \frac{|R_{m+n}|}{|R_m|} \text{ is } e^{-k} \tag{14}$$

for large $n$.

The ratio of the standard deviation of $r_n$ to $r_n$ is of the order of magnitude of $1/n^{\frac{1}{2}}$ times the ratio of the standard deviation of $1 - \Delta R_{m+1}/|R_m|$.
Since

$$1 - \frac{\Delta R_{m+1}}{|R_m|} \approx 1,$$

and the standard deviation has already been seen to be of the order of $k/n$, this ratio will also be of the order of $k/n$. Thus the standard deviation of $r_n$, which is even smaller by a factor of $1/n^{\frac{1}{2}}$, would introduce little chance that $r_n$ can be greatly in error.

Since $k$ is a number close to one, something like 20 such steps as are indicated in equation (14) will reduce the error as measured by $|R_m|$ to at most $10^{-6}$ of its original value. This is in general agreement with the usual experience in relaxation techniques [2]. Each such series of $n$ steps requires about as much computation as one iteration of the whole matrix if the process of selection of $c_{m+1}$ is ignored; this is of the order of $dn$ arithmetical manipulations where $d$ is a small whole number of the order of 10, depending on the number of terms of one row of $A$ which differ from zero. If the operation of "compare" which is necessary to find a maximum value for $c_m$, takes $e$ arithmetical operations, a total of about 20 $(e + d)n$ operations will be necessary to reduce $|R_m|$ to $10^{-6}$ of its original value. The presently contemplated methods for "compare" [4] would make $e$ equal to $n$ which makes the total about 20 $n^2$ for large $n$ regardless of the nature of $A$. This may be compared with the best results obtained by Frankel for iteration methods of about 20 $n^{3/2}$ for Poisson's equation and about 20 $n^2$ for the biharmonic equation. Thus, there is no saving in a relaxation method, which is actually more difficult to program, unless the problem is more complicated than the biharmonic equation. However, there is a possibility of making the "compare" operation more rapid by proper design, which would reduce the above estimate. From a purely heuristic viewpoint it is clear that by any method, at least $h\ n$ operations are necessary to evaluate $n$ points, where $h$ is a small number greater than one; the difference between this and the above estimate is completely taken up by the "compare" operation.

## References

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# THE STABILITY EQUATION WITH PERIODIC COEFFICIENTS*

By HIRSH COHEN (*Haifa, Israel*)

In a large number of physical problems involving periodic motion, dynamic stability considerations result in stability differential equations which have periodic coefficients. In particular, if the physical system is described by a non-linear second order ordinary differential equation, a second order equation of the Floquet type appears. That this is not an isolated case becomes apparent if one reviews the large volume of non-linear mechanics literature of the past few years. The problem to be discussed in this note is even more specialized than the one just introduced but the same review through the literature will reveal that it is an important case. This is the stability problem which results when the non-linear element has small effect on the system, i.e., when the resultant motion is near to the motion of the linearized system.

As an example consider the van der Pol equation

$$y'' - \epsilon(1 - y^2)y' + y = 0 \tag{1}$$

where primes refer to differentiation with respect to $t$. If $y$ is taken to be of 0 (1) then $\epsilon \ll 1$. The usual stability considerations involve the addition of a small (of order $\epsilon$) time-dependent function, $v(t)$, to an exact or approximate solution $y_0(t)$. On substitution into (1) of $y = y_0 + v(t)$, the equation of first order in $v(t)$ is

$$v'' - \epsilon(1 - y_0^2)v' + (1 + 2\epsilon y_0 y_0')v = 0. \tag{2}$$

If the solution is to be a periodic approximation to $y$, then $y_0$ is periodic and (2) represents an example of the general equation dealt with herein, namely

$$u'' + \epsilon p(t)u' + \epsilon\left(q_1(t) + \frac{1}{\epsilon}\right)u = 0, \tag{3}$$

where $u$ is the disturbance function being used to "test" some system, and $p(t)$ and $q_1(t)$ are periodic functions of period $2\pi/\omega$.

It can be seen immediately that the Mathieu equation is a special case of (3). Furthermore, it would appear useful to remove the first order term in (3) and thus reduce it to at worst a Hill equation. This may be done by the substitution

$$u = v(t) \exp\left[-\frac{\epsilon}{2}\int p(t)\,dt\right]. \tag{4}$$

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