# THE PLASTIC INSTABILITY OF PLATES*

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**1. Introduction.** A flow theory of the plastic instability of plates under edge stresses which can be used to provide information for structural design is not yet available. In this paper, an analysis of this problem, for rectangular plastes under uniform edge direct stresses, is given that provides information of a general nature only. In comparison with previous work by Handelman and Prager (1), Hopkins (2) and Pearson (3), the present paper exhibits both similarities and differences. In the first place, Refs. 1-3 and the present paper develop analyses within the framework of the elementary bending theory of plates and based upon a plastic flow theory relating to an idealized material whose mechanical behaviour is that of a work-hardening material obeying the Mises plasticity condition; further, in all these papers, the applied edge direct stress distributions are similar, always being uniform along an edge and taken to vanish for one pair of opposite edges in Refs. 1 and 3. In the second place, there is an implicit neglect of strain compatibility in the plate middle surface in Refs. 1-3. In brief, in comparison with the analyses of Refs. 1-3, the important features of the present analysis are that strain compatibility of the plate middle surface is included and that the development of the stress-rate/strain-rate relationships is more incisive. The present paper is based upon Ref. 4, and certain parts of the analysis developed there are here either curtailed or omitted for conciseness.

**2. Notation.** The length, breadth and thickness of the plate are respectively $a$, $b$ and $2h$. The tensor notation employed in Section 3 is described there. In Section 4, the origin $O$ is a point common to an edge of length $b$ and the middle surface; the $x$- and $y$-axes are in the middle surface, and parallel to the plate edges; and the $z$-axis is such that the frame of reference $O(x, y, z)$ is right-handed. The displacement of a point during instability has the components $hu$, $hv$ and $hw$ respectively parallel to the $x$-, $y$-and $z$-axes. The notation for strains, stresses and plate resultants is standard except that reduced stresses, i.e. stresses divided by Young's modulus $E_0$, are used. The tangent modulus of elasticity is $E$, and Poisson's ratio (in the elastic range) is $\nu(0 \leq \nu \leq \frac{1}{2})$. The analysis is simplified with the use of the non-dimensional coordinates $\xi$, $\eta$ and $\zeta$ defined by

$$\xi = x/a, \qquad \eta = y/a, \qquad \zeta = z/h, \tag{1}$$

and then, with an appropriate choice of the co-ordinate system, the origin now being taken at a corner of the plate, the unstrained plate occupies the region defined by

$$0 \leq \xi \leq 1, \qquad 0 \leq \eta \leq b/a, \qquad -1 \leq \zeta \leq 1. \tag{2}$$

Immediately before normal displacements occur, the plate is subject to the state of uniform normal edge stress defined by

$$\left.\begin{aligned} \sigma_x &= -\sigma_1(\xi = 0,\, 1; \qquad 0 \leq \eta \leq b/a; \qquad -1 \leq \zeta \leq 1), \\ \sigma_y &= -\sigma_2(\eta = 0,\, b/a; \qquad 0 \leq \xi \leq 1; \qquad -1 \leq \zeta \leq 1), \end{aligned}\right\} \tag{3}$$

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where at least one of $\sigma_1$ and $\sigma_2$ is necessarily positive if normal displacement of the middle surface is to be possible. The following abbreviations are used:

$$
\left.
\begin{aligned}
\alpha &= a^2/b^2, \qquad \beta = h^2/a^2, \\
\sigma_1' &= 3(1-\nu^2)\sigma_1/\beta, \qquad \sigma_2' = 3(1-\nu^2)\sigma_2/\beta, \\
r_1 &= (2\sigma_1 - \sigma_2)/(\sigma_1 - 2\sigma_2), \qquad r_2 = (2\sigma_2 - \sigma_1)/(\sigma_2 - 2\sigma_1), \\
L &= -\left(1 - \frac{\nu}{r_1}\right)\left(1 - \frac{\nu}{r_2}\right) + \lambda_1\left(1 - \frac{\nu}{r_1}\right) + \lambda_2\left(1 - \frac{\nu}{r_2}\right) \\
&= (1-\nu^2) + (\lambda_1 - 1)\{1 + 1/r_1^2 - 2\nu/r_1\} > 0, \\
c_1 &= (\lambda_1 - 1)/L(2\sigma_1 - \sigma_2), \qquad c_2 = (\lambda_2 - 1)/L(2\sigma_2 - \sigma_1), \\
M &= 1 - 2L\Big/\left\{\left(1 - \frac{\nu}{r_1}\right)(\lambda_1 - 1) + \left(1 - \frac{\nu}{r_2}\right)(\lambda_2 - 1)\right\} \\
&= -1 - 2(1-\nu^2)/(\lambda_1 - 1)(1 + 1/r_1^2 - 2\nu/r_1) \leq -1, \\
N &= \{(1 - \nu/r_1)(\lambda_1 - 1) + (1 - \nu/r_2)(\lambda_2 - 1)\}/4(1-\nu^2) \\
&= (\lambda_1 - 1)(1 + 1/r_1^2 - 2\nu/r_1)/4(1-\nu^2) \geq 0, \\
d^* &= \tfrac{1}{4}(2 + \zeta_0)(1 - \zeta_0)^2, \\
\delta_1 &= -d^*(\lambda_1 - 1)(1 - \nu/r_1)/L, \qquad \delta_2 = -d^*(\lambda_2 - 1)(1 - \nu/r_2)/L, \\
D_{11} &= 1 + (1 - \nu/r_1)\delta_1, \\
D_{22} &= 1 + (1 - \nu/r_2)\delta_2, \\
2D_{12} &= 2 + (\nu - 1/r_1)\delta_1 + (\nu - 1/r_2)\delta_2 .
\end{aligned}
\right\} \tag{4}
$$

In the analysis it is convenient to treat with rates-of-change of certain mathematical quantities rather than with their small changes, and primes generally denote time differentation.

**3. Relationships Between the Stress- and Strain-Rates.** In this Section, following Refs. (5) and (6), relationships are determined between stress- and strain-rates for a special type of work-hardening material; these relationships are developed within the framework of the elementary bending theory of thin plates and are appropriate to a rectangular plate, loaded initially only by uniform direct stresses along its edges, and in a configuration close to its initially flat configuration. Tensor notation is used in this Section; thus, if $Ox_i$ denotes a right-handed rectangular cartesian frame of reference, then $\sigma_{ij}$ denotes the stress tensor and $\epsilon_{ij}'$ denotes the strain-rate tensor corresponding to the stress-rate tensor $\sigma_{ij}'$, where Latin suffixes have the values 1, 2 and 3, and the repeated suffix convention for summation is employed. For simplicity, reduced stresses and stress-rates, i.e. stresses and stress-rates divided by Young's modulus $E_0$, are employed.

The plastic strain-rate $\epsilon_{ij}'^{p}$ is defined by the equation

$$\epsilon_{ij}'^{p} = \epsilon_{ij}' - \epsilon_{ij}'^{e} , \tag{5}$$

where the elastic strain-rate $\epsilon_{ij}'^{e}$ is related to the stress-rate $\sigma_{ij}'$ through Hooke's law, viz.

$$\epsilon_{ij}'^{e} = (1 + \nu)\sigma_{ij}' - \nu\sigma_{kk}'\delta_{ij} . \tag{6}$$

Then it is proved in Ref. 6 that the plastic strain-rate is given by the equation

$$\epsilon_{ij}'^{p} = G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' , \tag{7}$$

where $G$ is a non-zero scalar that may depend upon the current stress- and strain-states, together with their histories, but does not involve the current stress-rate, and where $f$ is the loading function. Thus, from Eqs. (5)-(7), the stress-rate/strain-rate relationships are given by the equations,

$$\left.\begin{aligned} \epsilon_{ij}' &= (1 + \nu)\sigma_{ij}' - \nu\sigma_{kk}'\delta_{ij} + G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' , \qquad && \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' > 0, \\ \epsilon_{ij}' &= (1 + \nu)\sigma_{ij}' - \nu\sigma_{kk}'\delta_{ij} , && \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' < 0. \end{aligned}\right\} \tag{8}$$

It is assumed now that the plastic strain-rates are incompressible so that

$$\epsilon_{kk}'^{p} = 0, \tag{9}$$

and from Eq. (7) it follows that

$$\frac{\partial f}{\partial \sigma_{kk}} \equiv \frac{\partial f}{\partial \sigma_{11}} + \frac{\partial f}{\partial \sigma_{22}} + \frac{\partial f}{\partial \sigma_{33}} = 0, \tag{10}$$

or, equivalently,

$$\frac{\partial f}{\partial \sigma} = 0 \qquad \text{where} \qquad \sigma = \frac{1}{3} \sigma_{kk} , \tag{11}$$

and hence the loading function is independent of $\sigma$ and therefore depends only upon $\sigma_{ij}^{*}$ where

$$\sigma_{ij}^{*} = \sigma_{ij} - \frac{1}{3} \sigma_{kk}\delta_{ij} . \tag{12}$$

If the co-ordinate axes $0x_i$ are suitably oriented then the initial state of stress in the plate, i.e. just before normal displacements occur, is represented by the tensor

$$\sigma_{ij} \equiv \begin{Vmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{Vmatrix} . \tag{13}$$

In the present connection, i.e. within the framework of the elementary bending theory of thin plates, the stress-rate tensor to which explicit attention is given when normal displacements first occur is represented by

$$\sigma_{ij}' \equiv \begin{Vmatrix} \sigma_{11}' & \sigma_{12}' & 0 \\ \sigma_{21}' & \sigma_{22}' & 0 \\ 0 & 0 & 0 \end{Vmatrix} . \tag{14}$$

Therefore, from Eqs. (8) and (9), the stress-rate/strain-rate relationships, significant in the present connection, are given by the equations,

$$\left.\begin{array}{l}
\left.\begin{aligned}
\epsilon'_{\alpha\beta} &= (1+\nu)\sigma'_{\alpha\beta} - \nu\sigma'_{\gamma\gamma}\delta_{\alpha\beta} + G\frac{\partial f}{\partial\sigma_{\alpha\beta}}\frac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta}, \\
\epsilon'_{33} &= -\nu\sigma'_{\gamma\gamma} - G\left(\frac{\partial f}{\partial\sigma_{11}} + \frac{\partial f}{\partial\sigma_{22}}\right)\frac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta}.
\end{aligned}\right\} \frac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta} > 0, \\
\left.\begin{aligned}
\epsilon'_{\alpha\beta} &= (1+\nu)\sigma'_{\alpha\beta} - \nu\sigma'_{\gamma\gamma}\delta_{\alpha\beta}, \\
\epsilon'_{33} &= -\nu\sigma'_{\gamma\gamma},
\end{aligned}\right\} \frac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta} < 0,
\end{array}\right\} \tag{15}$$

where Greek suffixes have the values 1 and 2, and the repeated suffix convention for summation is employed. It is reasonable to assume that the direct stress-rates produce zero or negligible shear strain-rates, and then, from Eqs. (15),

$$\frac{\partial f}{\partial\sigma_{12}} = 0 \qquad \text{for} \qquad \sigma_{12} = \sigma_{33} = \sigma_{13} = 0. \tag{16}$$

Eqs. (15) now simplify to the equations,

$$\left.\begin{array}{l}
\left.\begin{aligned}
\epsilon'_{11} &= \sigma'_{11} - \nu\sigma'_{22} \\
&\qquad + G\frac{\partial f}{\partial\sigma_{11}}\left(\frac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \frac{\partial f}{\partial\sigma_{22}}\sigma'_{22}\right), \\
\epsilon'_{12} &= (1+\nu)\sigma'_{12}, \\
\epsilon'_{22} &= -\nu\sigma'_{11} + \sigma'_{22} \\
&\qquad + G\frac{\partial f}{\partial\sigma_{22}}\left(\frac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \frac{\partial f}{\partial\sigma_{22}}\sigma'_{22}\right), \\
\epsilon'_{33} &= -\nu(\sigma'_{11} + \sigma'_{22}) \\
&\qquad - G\left(\frac{\partial f}{\partial\sigma_{11}} + \frac{\partial f}{\partial\sigma_{22}}\right)\left(\frac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \frac{\partial f}{\partial\sigma_{22}}\sigma'_{22}\right),
\end{aligned}\right\} \frac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \frac{\partial f}{\partial\sigma_{22}}\sigma'_{22} > 0, \\
\left.\begin{aligned}
\epsilon'_{11} &= \sigma'_{11} - \nu\sigma'_{22}, \\
\epsilon'_{12} &= (1+\nu)\sigma'_{12}, \\
\epsilon'_{22} &= -\nu\sigma'_{11} + \sigma'_{22}, \\
\epsilon'_{33} &= -\nu(\sigma'_{11} + \sigma'_{22}),
\end{aligned}\right\} \frac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \frac{\partial f}{\partial\sigma_{22}}\sigma'_{22} < 0.
\end{array}\right\} \tag{17}$$

Now define two scalars $\lambda_1$ and $\lambda_2$ through the equations

$$\left.\begin{aligned} \lambda_1 &= \epsilon'_{11}/\sigma'_{11}, \quad \sigma'_{22} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{11}}\sigma'_{11} > 0, \\ \lambda_2 &= \epsilon'_{22}/\sigma'_{22}, \quad \sigma'_{11} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{22}}\sigma'_{22} > 0, \end{aligned}\right\} \tag{18}$$

and then, from Eqs. (17),

$$\lambda_1 = 1 + G\left(\frac{\partial f}{\partial \sigma_{11}}\right)^2, \qquad \lambda_2 = 1 + G\left(\frac{\partial f}{\partial \sigma_{22}}\right)^2, \tag{19}$$

and hence

$$G^{1/2}\frac{\partial f}{\partial \sigma_{11}} = \pm(\lambda_1 - 1)^{1/2}, \qquad G^{1/2}\frac{\partial f}{\partial \sigma_{22}} = \pm(\lambda_2 - 1)^{1/2}, \tag{20}$$

where, in the first equation, the positive and negative signs correspond respectively to loading and unloading in the special case $\sigma'_{11} > 0$ with $\sigma'_{22} = 0$, and similarly for the second of these equations. Therefore, from Eqs. (17) and (20), it is possible to formulate stress-rate/strain-rate relationships that involve the two scalars $\lambda_1$ and $\lambda_2$ whose numerical values could be determined from experimental data. Note that

$$(\lambda_1 - 1)/(\lambda_2 - 1) = \left(\frac{\partial f}{\partial \sigma_{11}}\Big/\frac{\partial f}{\partial \sigma_{22}}\right)^2. \tag{21}$$

The simplest possible assumption for $f$ is that the Mises plasticity condition applies, viz.

$$f = \frac{1}{2}\sigma^*_{ij}\sigma^*_{ij} = k^2, \tag{22}$$

where the strain-history enters only through the single parameter $k$. Loading occurs only if

$$f' = \sigma^*_{ij}\sigma'^*_{ij} > 0, \tag{23}$$

and in the present connection the criterion is therefore that

$$\begin{aligned} &\left\{\sigma_{11} - \frac{1}{3}(\sigma_{11} + \sigma_{22})\right\}\left\{\sigma'_{11} - \frac{1}{3}(\sigma'_{11} + \sigma'_{22})\right\} \\ &\quad + \left\{\sigma_{22} - \frac{1}{3}(\sigma_{11} + \sigma_{22})\right\}\left\{\sigma'_{22} - \frac{1}{3}(\sigma'_{11} + \sigma'_{22})\right\} + \frac{1}{9}(\sigma_{11} + \sigma_{22})(\sigma'_{11} + \sigma'_{22}) > 0, \end{aligned} \tag{24}$$

i.e.,

$$f' = \frac{1}{3}\{(2\sigma_{11} - \sigma_{22})\sigma'_{11} + (2\sigma_{22} - \sigma_{11})\sigma'_{22}\} > 0. \tag{25}$$

Thus

$$\frac{\partial f}{\partial \sigma_{11}} = \frac{1}{3}(2\sigma_{11} - \sigma_{22}), \qquad \frac{\partial f}{\partial \sigma_{22}} = \frac{1}{3}(2\sigma_{22} - \sigma_{11}), \tag{26}$$

and accordingly, from Eq. (21),

$$(\lambda_1 - 1)/(\lambda_2 - 1) = \{(2\sigma_{11} - \sigma_{22})/(2\sigma_{22} - \sigma_{11})\}^2. \tag{27}$$

From Eqs. (19) and (26),

$$G = 9(\lambda_1 - 1)/(2\sigma_{11} - \sigma_{22})^2 = 9(\lambda_2 - 1)/(2\sigma_{22} - \sigma_{11})^2, \tag{28}$$

and G is calculable if either $\lambda_1$ or $\lambda_2$ is known from experimental data. It is possible therefore to formulate the stress-rate/strain-rate relationships in terms of a single scalar to be determined necessarily from experimental data; these relationships are

$$\left.\begin{array}{l}
\left.\begin{aligned}
\epsilon'_{11} &= \lambda_1\sigma'_{11} - \left\{\nu + \frac{1}{R_1}(\lambda_1 - 1)\right\}\sigma'_{22}\,,\\
\epsilon'_{12} &= (1 + \nu)\sigma'_{12}\,,\\
\epsilon'_{22} &= -\left\{\nu + \frac{1}{R_2}(\lambda_2 - 1)\right\}\sigma'_{11} + \lambda_2\sigma'_{22}\,,\\
\epsilon'_{33} &= -\left\{\nu + (\lambda_1 - 1) - \frac{1}{R_2}(\lambda_2 - 1)\right\}\sigma'_{11}\\
&\quad -\left\{\nu + (\lambda_2 - 1) - \frac{1}{R_1}(\lambda_1 - 1)\right\}\sigma'_{22}\,,
\end{aligned}\right\}(2\sigma_{11} - \sigma_{22})\sigma'_{11} + (2\sigma_{22} - \sigma_{11})\sigma'_{22} > 0,\\[2ex]
\left.\begin{aligned}
\epsilon'_{11} &= \sigma'_{11} - \nu\sigma'_{22}\,,\\
\epsilon'_{12} &= (1 + \nu)\sigma'_{12}\,,\\
\epsilon'_{22} &= -\nu\sigma'_{11} + \sigma'_{22}\,,\\
\epsilon'_{33} &= -\nu(\sigma'_{11} + \sigma'_{22}),
\end{aligned}\right\}(2\sigma_{11} - \sigma_{22})\sigma'_{11} + (2\sigma_{22} - \sigma_{11})\sigma'_{22} < 0,
\end{array}\right\} \tag{29}$$

where

$$R_1 = (2\sigma_{11} - \sigma_{22})/(\sigma_{11} - 2\sigma_{22}), \qquad R_2 = (2\sigma_{22} - \sigma_{11})/(\sigma_{22} - 2\sigma_{11}), \tag{30}$$

and $\lambda_1$ and $\lambda_2$ are related through Eq. (27). These stress-rate/strain-rate relationships agree with those obtained in Refs. 1 and 2. However, in Ref. 2, the result Eq. (27) could be deduced only for the particular case discussed in Ref. 1, viz. $\sigma_{22}$ equal to zero, and the result is then

$$\lambda_2 = \frac{1}{4}(\lambda_1 + 3), \tag{31}$$

where $\lambda_1$ is interpreted simply as $E_0/E$.

In the next Section, it is more convenient to use an engineering notation for stresses and strains, and accordingly Eqs. (29) are rewritten now in the form,

$$\left.\begin{array}{l}
\left.\begin{aligned}
\epsilon_x' &= \lambda_1\sigma_x' - \left\{\nu + \frac{1}{r_1}(\lambda_1 - 1)\right\}\sigma_y', \\
\gamma_{xy}' &= 2(1+\nu)\tau_{xy}', \\
\epsilon_y' &= -\left\{\nu + \frac{1}{r_2}(\lambda_2 - 1)\right\}\sigma_x' + \lambda_2\sigma_y', \\
\epsilon_z' &= -\left\{\nu + (\lambda_1 - 1) - \frac{1}{r_2}(\lambda_2 - 1)\right\}\sigma_x' \\
&\quad -\left\{\nu + (\lambda_2 - 1) - \frac{1}{r_1}(\lambda_1 - 1)\right\}\sigma_y',
\end{aligned}\right\}(2\sigma_1 - \sigma_2)\sigma_x' + (2\sigma_2 - \sigma_1)\sigma_y' < 0, \\
\left.\begin{aligned}
\epsilon_x' &= \sigma_x' - \nu\sigma_y', \\
\gamma_{xy}' &= 2(1+\nu)\tau_{xy}', \\
\epsilon_y' &= -\nu\sigma_x' + \sigma_y', \\
\epsilon_z' &= -\nu(\sigma_x' + \sigma_y'),
\end{aligned}\right\}(2\sigma_1 - \sigma_2)\sigma_x' + (2\sigma_2 - \sigma_1)\sigma_y' > 0.
\end{array}\right\} \tag{32}$$

Let the rate at which work is done by existing stresses on the change of shape of an element be $W'$ per unit volume so that

$$W' = E_0\sigma_{ij}\epsilon_{ij}'^{*}. \tag{33}$$

It may be proved from Eqs. (8) and (22) that

$$W' = \left\{\begin{array}{ll} [(1+\nu) + 2Gk^2]E_0 f', & f' > 0, \\ (1+\nu)E_0 f', & f' < 0. \end{array}\right\} \tag{34}$$

Thus the loading condition that $f'$ is positive is interpreted that $W'$ is positive; the neutral condition that $f'$ is zero is interpreted that $W'$ is zero; and the unloading condition that $f'$ is negative is interpreted that $W'$ is negative. From Eqs. (13) and (32) the loading and unloading criteria are now written in the form

$$\left.\begin{array}{c} \{(2\sigma_2 - \sigma_1)\nu + (2\sigma_1 - \sigma_2)\}\epsilon_x' + \{(2\sigma_1 - \sigma_2)\nu + (2\sigma_2 - \sigma_1)\}\epsilon_y' \lesseqgtr 0, \\ W' \gtrless 0, \end{array}\right\} \tag{35}$$

respectively.

**4. The Fundamental Equations.** In this Section, the fundamental equations are developed on the basis of the stress-rate/strain-rate relationships in Eqs. (32). It is assumed, exactly as in elastic thin plate theory, that plate elements originally normal to the middle surface remain normal to the displaced middle surface. Then

$$\left.\begin{aligned} \epsilon_x' &= \epsilon_1' - \beta\zeta w'_{,\xi\xi} , \\ \epsilon_y' &= \epsilon_2' - \beta\zeta w'_{,\eta\eta} , \\ \gamma_{xy}' &= \gamma' - 2\beta\zeta w'_{,\xi\eta} , \end{aligned}\right\} \tag{36}$$

where $\epsilon_1'$, $\epsilon_2'$ and $\gamma'$ are strain-rates in the middle surface. Here, and also elsewhere in this paper, suffixes $\xi$ and $\eta$ (and also $n$ and $s$) preceded by a comma denote partial differential coefficients. From condition (35) and Eqs. (36), the loading criterion is

$$\zeta \gtrless \zeta_0 \qquad \text{if} \qquad K' \gtrless 0 \tag{37}$$

respectively, where

$$\left.\begin{aligned} \beta\zeta_0 K' &= (2\sigma_1 - \sigma_2)\{(1 - \nu/r_1)\epsilon_1' + (\nu - 1/r_1)\epsilon_2'\}, \\ K' &= (2\sigma_1 - \sigma_2)\{(1 - \nu/r_1)w'_{,\xi\xi} + (\nu - 1/r_1)w'_{,\eta\eta}\}. \end{aligned}\right\} \tag{38}$$

It is necessary always that $-1 \leq \zeta_0 \leq 1$; if $-1 < \zeta_0 < 1$, then the surface with equation $\zeta = \zeta_0$ separates loading and unloading regions, and on this surface $W' = 0$; $\zeta_0 = \pm 1$ corresponds to either loading or unloading over the entire plate thickness, and on the (plate) surface $\zeta = \pm 1$, in general $W' \neq 0$, and therefore this surface is not in this sense a boundary to a loading or an unloading region. It is seen by setting either $w'_{,\xi\xi}$ or $w'_{,\eta\eta}$ equal to zero that the sign of $K'$ is an indication of the sense of bending.

The resultant force-rates $N_x'$, $N_y'$ and $N_{xy}'$ are defined by the equations

$$(N_x' , N_y' , N_{xy}')/h = \int_{-1}^{1} (\sigma_x' , \sigma_y' , \tau_{xy}')\, d\zeta, \tag{39}$$

and therefore

$$(N_x' - \nu N_y' , N_y' - \nu N_x')/h = \int_{-1}^{1} (\sigma_x' - \nu\sigma_y' , \sigma_y' - \nu\sigma_x')\, d\zeta. \tag{40}$$

From Eqs. (32) and (38) and conditions (37),

$$\left.\begin{aligned} \sigma_x' - \nu\sigma_y' &= \epsilon_x' + \beta c_1(\zeta - \zeta_0)K', \\ \sigma_y' - \nu\sigma_x' &= \epsilon_y' + \beta c_2(\zeta - \zeta_0)K', \\ \tau_{xy}' &= \gamma_{xy}'/2(1 + \nu), \end{aligned}\right\} \qquad (W' > 0), \tag{41}$$

and

$$\left.\begin{aligned} \sigma_x' - \nu\sigma_y' &= \epsilon_x' , \\ \sigma_y' - \nu\sigma_x' &= \epsilon_y' , \\ \tau_{xy}' &= \gamma_{xy}'/2(1 + \nu), \end{aligned}\right\} \qquad (W' < 0). \tag{42}$$

Conditions (37) show that there are two cases to be considered and the analysis now proceeds differently according as $K' > 0$ (Case $A$) or $K' < 0$ (Case $B$).

*Case A*. Here $K' > 0$, and then $(1 \geq \zeta > \zeta_0)$ and $(\zeta_0 > \zeta \geq -1)$ are respectively the loading and unloading regions.

From Eqs. (36) and (39)-(42),

$$\left.\begin{aligned} (N_x' - \nu N_y')/2h &= \epsilon_1' + \tfrac{1}{4}\beta c_1(1 - \zeta_0)^2 K', \\ (N_y' - \nu N_x')/2h &= \epsilon_2' + \tfrac{1}{4}\beta c_2(1 - \zeta_0)^2 K', \\ N_{xy}'/2h &= \gamma'/2(1 + \nu), \end{aligned}\right\} \tag{43}$$

and therefore

$$-\tfrac{1}{4}\beta(1 - \zeta_0)^2 K' = \{\epsilon_1' - (N_x' - \nu N_y')/2h\}/c_1 = \{\epsilon_2' - (N_y' - \nu N_x')/2h\}/c_2 . \tag{44}$$

From previous remarks, and from Eqs. (38.1) and (44), $\zeta_0 = \zeta_1$ if $-1 \leq \zeta_1 \leq 1$ and $\zeta_0 = \pm 1$ otherwise where $\zeta_1 = \zeta^*(P)$ is defined by the equation

$$\zeta^{*2} - 2M\zeta^* + P = 0, \tag{45}$$

i.e.

$$\zeta^* = M \pm (M^2 - P)^{1/2}, \tag{46}$$

and

$$\left.\begin{aligned} P = 1 - 4L\{&(2\sigma_1 - \sigma_2)(1 - \nu/r_1)p_1 + (2\sigma_2 - \sigma_1)(1 - \nu/r_2)p_2\} \\ &\div \{(\lambda_1 - 1)(1 - \nu/r_1) + (\lambda_2 - 1)(1 - \nu/r_2)\}, \\ p_1 = (N_x' &- \nu N_y')/2h\beta K', \\ p_2 = (N_y' &- \nu N_x')/2h\beta K'. \end{aligned}\right\} \tag{47}$$

If $-1 \leq \zeta^*(P) \leq 1$, where $\zeta^*(P)$ is defined by Eq. (46), then $\zeta_0 = \zeta^*(P)$ and the surface with equation $\zeta = \zeta_0 = \zeta^*(P)$ separates loading and unloading regions of the plate. If such a surface of separation exists in the immediate neighbourhood of some particular point of the middle surface, then, at this point, it is necessary first that

$$M^2 \geq P \tag{48}$$

and second that it is then possible to choose the sign in Eq. (46) so that

$$-1 \leq \zeta^*(P) \leq 1; \tag{49}$$

if either of the conditions (48) and (49) is violated, then loading or unloading occurs over the whole of the plate thickness in the immediate neighbourhood of this point, and then $\zeta_0 = \pm 1$. It is therefore not impossible for loading (or unloading) to occur over the entire thickness of part of the plate and a varying degree of loading (or unloading) to occur over the remaining part. Note that both roots of Eq. (45) may satisfy $-1 \leq \zeta^*(P) \leq 1$ in which case there are two possible solutions.

The resultant bending and twisting moment-rates $M_x'$, $M_y'$ and $M_{xy}'(= -M_{yx}')$ are defined by the equations

$$(M_x', M_y', M_{xy}')/h^2 = \int_{-1}^{1} (\sigma_x', \sigma_y', -\tau_{xy}')\zeta \, d\zeta, \tag{50}$$

and therefore

$$(M_x' - \nu M_y' \, , \, M_y' - \nu M_x')/h^2 = \int_{-1}^{1} (\sigma_x' - \nu\sigma_y' \, , \, \sigma_y' - \nu\sigma_x')\zeta \, d\zeta. \tag{51}$$

From Eqs. (36), (41), (42), (50) and (51),

$$\left.\begin{aligned} M_x' - \nu M_y' &= -\tfrac{2}{3}h^2\beta(w'_{,\xi\xi} - dc_1K'), \\ M_y' - \nu M_x' &= -\tfrac{2}{3}h^2\beta(w'_{,\eta\eta} - dc_2K'), \\ M_{xy}' = -M_{yx}' &= \{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\}(1 - \nu)w'_{,\xi\eta} \, , \end{aligned}\right\} \tag{52}$$

where $d = d^*(\zeta_0)$, and therefore

$$\left.\begin{aligned} -M_x'/\{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\} &= \{1 + \delta_1(1 - \nu/r_1)\}w'_{,\xi\xi} + \{\nu + \delta_2(\nu - 1/r_2)\}w'_{,\eta\eta} \, , \\ -M_y'/\{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\} &= \{\nu + \delta_1(\nu - 1/r_1)\}w'_{,\xi\xi} + \{1 + \delta_2(1 - \nu/r_2)\}w'_{,\eta\eta} \, , \\ M_{xy}'/\{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\} &= (1 - \nu)w'_{,\xi\eta} \, . \end{aligned}\right\} \tag{53}$$

The resultant total vertical shear force-rates $V_x'$ and $V_y'$ are defined by the equations

$$\left.\begin{aligned} aV_x' &= M'_{x,\xi} - 2M'_{xy,\eta} \, , \\ aV_y' &= M'_{y,\eta} - 2M'_{xy,\xi} \, , \end{aligned}\right\} \tag{54}$$

and therefore from Eqs. (53)

$$\left.\begin{aligned} -V_x'/\{\tfrac{2}{3}a\beta^2/(1 - \nu^2)\} &= \frac{\partial}{\partial\xi}[\{1 + \delta_1(1 - \nu/r_1)\}w'_{,\xi\xi} + \{(2 - \nu) + \delta_2(\nu - 1/r_2)\}w'_{,\eta\eta}], \\ -V_y'/\{\tfrac{2}{3}a\beta^2/(1 - \nu^2)\} &= \frac{\partial}{\partial\eta}[\{1 + \delta_2(1 - \nu/r_2)\}w'_{,\eta\eta} + \{(2 - \nu) + \delta_1(\nu - 1/r_1)\}w'_{,\xi\xi}]. \end{aligned}\right\} \tag{55}$$

Now consider the equilibrium of the plate when small normal displacements have occurred. The equations of equilibrium (7, § 57) of the plate, for small normal displacements of its middle surface, are

$$\left.\begin{aligned} M_{x,\xi\xi} - 2M_{xy,\xi\eta} + M_{y,\eta\eta} &= -h(N_x w_{,\xi\xi} + 2N_{xy}w_{,\xi\eta} + N_y w_{,\eta\eta}), \\ N_{x,\xi} + N_{xy,\eta} &= 0, \\ N_{xy,\xi} + N_{y,\eta} &= 0; \end{aligned}\right\} \tag{56}$$

partial differentiation of Eqs. (56) with respect to time shows that

$$\left.\begin{aligned} M'_{x,\xi\xi} - 2M'_{xy,\xi\eta} + M'_{y,\eta\eta} &= 2h^2(\sigma_1 w'_{,\xi\xi} + \sigma_2 w'_{,\eta\eta}), \\ N'_{x,\xi} + N'_{xy,\eta} &= 0, \\ N'_{xy,\xi} + N'_{y,\eta} &= 0, \end{aligned}\right\} \tag{57}$$

immediately following the incidence of normal displacements because immediately before these occur $w$ and $N_{xy}$ are identically zero. From Eqs. (53) and (57.1),

$$\nabla_1^4 w' + (1 - \nu/r_1) \frac{\partial^2}{\partial \xi^2} (\delta_1 w'_{,\xi\xi}) + (\nu - 1/r_2) \frac{\partial^2}{\partial \xi^2} (\delta_2 w'_{,\eta\eta})$$

$$+ (\nu - 1/r_1) \frac{\partial^2}{\partial \eta^2} (\delta_1 w'_{,\xi\xi}) + (1 - \nu/r_2) \frac{\partial^2}{\partial \eta^2} (\delta_2 w'_{,\eta\eta}) = -(\sigma_1' w'_{,\xi\xi} + \sigma_2' w'_{,\eta\eta}), \quad (58)$$

where

$$\nabla_1^4 \equiv \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^2. \quad (59)$$

Equations (57.2) and (57.3) are satisfied identically through the introduction of a stress-rate function $\varphi'$ that is related to the resultant force-rates by the equations,

$$N_x'/2h = \varphi'_{,\eta\eta}, \qquad N_y'/2h = \varphi'_{,\xi\xi}, \qquad N_{xy}'/2h = -\varphi'_{,\xi\eta}. \quad (60)$$

The strain-rates $\epsilon_x'$, $\epsilon_y'$ and $\gamma_{xy}'$ are given by the equations

$$\epsilon_x' = \beta^{1/2} u'_{,\xi}, \qquad \epsilon_y' = \beta^{1/2} v'_{,\eta}, \qquad \gamma_{xy}' = \beta^{1/2}(u'_{,\eta} + v'_{,\xi}), \quad (61)$$

and are compatible only if

$$\epsilon'_{x,\eta\eta} + \epsilon'_{y,\xi\xi} - \gamma'_{xy,\xi\eta} = 0, \quad (62)$$

i.e. from Eqs. (36),

$$\epsilon'_{1,\eta\eta} + \epsilon'_{2,\xi\xi} - \gamma'_{,\xi\eta} = 0, \quad (63)$$

which is an equation that involves the middle surface strain-rates only. From Eqs. (44) and (60),

$$\left.\begin{aligned} \epsilon_1' &= \varphi'_{,\eta\eta} - \nu\varphi'_{,\xi\xi} - \tfrac{1}{4}\beta c_1(1 - \zeta_0)^2 K', \\ \epsilon_2' &= \varphi'_{,\xi\xi} - \nu\varphi'_{,\eta\eta} - \tfrac{1}{4}\beta c_2(1 - \zeta_0)^2 K', \\ \gamma' &= -2(1 + \nu)\varphi'_{,\xi\eta}, \end{aligned}\right\} \quad (64)$$

and therefore Eq. (63) is

$$\nabla_1^4 \varphi' = \frac{1}{4} \beta\left(c_2 \frac{\partial^2}{\partial \xi^2} + c_1 \frac{\partial^2}{\partial \eta^2}\right)\{(1 - \zeta_0)^2 K'\}. \quad (65)$$

*Case B*. Here $K' < 0$, and then $(1 \geq \zeta > \zeta_0)$ and $(\zeta_0 > \zeta \geq -1)$ are respectively the loading and unloading regions. It is straightforward to show that the analysis of Case $B$ is deducible from the analysis of Case $A$ if all quantities are now referred to the coordinate system $O'(\xi', \eta', \zeta')$ where

$$\xi' = \text{const.} - \xi, \qquad \eta = \text{const.} - \eta, \qquad \zeta' = -\zeta, \quad (66)$$

the values of the constants being irrelevant. Full details are given in Ref. 4, and here only the results are stated. The equations that correspond to Eqs. (53), (55) and (58) are identical in form but now $d = d^*(-\zeta_0)$ and $\zeta_1 = -\zeta^*(-p_1, -p_2)$; the equations that correspond to Eqs. (60) are identical in form; and the equation that corresponds to Eq. (65) is

$$\nabla_1^4 \varphi' = -\frac{1}{4} \beta\left(c_2 \frac{\partial^2}{\partial \xi^2} + c_1 \frac{\partial^2}{\partial \eta^2}\right)\{(1 + \zeta_0)^2 K'\} \quad (67)$$

where again $\zeta_1 = -\zeta^*(-p_1, -p_2)$. The convention is now adopted that the above changes are implied if $K' < 0$, and then Eqs. (53), (55,) (58), (60) and (65) apply unrestrictedly.

The fundamental mathematical problem is therefore to solve the non-linear, dependent, fourth order, homogeneous, partial differential equations (58) and (65) subject to boundary conditions on $w'$ and $\varphi'$. These boundary conditions depend upon respectively the plate edge support and the applied resultant force-rates, and, in general, are not arbitrarily prescribed. If the middle surface is sub-divided into two or more regions by a curve $c$, say with normal direction $n$, on which $K' = 0$, then it is necessary on physical grounds that $w'$ and $\varphi'$, together with their first three partial differential coefficients with respect to $n$, are continuous on $c$; but, from Eqs. (58) and (60), $w'_{,nnnn}$ and $\varphi'_{,nnnn}$ are not, in general, continuous on $c$ and therefore in general the analytical form of the solution is not the same on either side of $c$.

Now consider the application of the analysis to two particular classes of problem. These are often designated in the published literature as the von Kármán and Shanley problems; in this paper they are designated briefly as problems $A$ and $B$, respectively.

Let $n$ and $s$ denote directions respectively normal to, and along, a plate edge.

*Problem A*. Here it is postulated that the changes of the applied edge stresses at instability are zero. Then, by definition of the problem,

$$\varphi'_{,ss} = \varphi'_{,ns} = 0 \tag{68}$$

along an edge, but note that, because Eq. (65) is not homogeneous in $\varphi'$, Eqs. (68) do not imply that $\varphi'$ can be taken as identically zero. The boundary conditions on $w'$ depend upon the plate edge support; for example, if the edges are either clamped or simply-supported (which are important practical cases) then the boundary conditions on $w'$ are respectively either

$$w' = w'_{,n} = 0 \tag{69}$$

or, from Eqs. (53),

$$w' = w'_{,nn} = 0$$

along an edge. The solution of Eqs. (58) and (65) subject to the boundary conditions (68) and, for example, (69) is clearly, in general, an extremely difficult problem. If the instability stresses do not greatly exceed the elastic limit, then it is not unreasonable to approximate Eq. (65) by the equation

$$\nabla_1^4\varphi' = 0. \tag{70}$$

This approximation is made implicitly in Refs. 1 and 2, and results in a considerable simplification of the problem. The appropriate solution of Eq. (70) subject to the conditions (68) is

$$\varphi' = 0. \tag{71}$$

Then, from Eqs. (46) and (47), remembering that $M$ is not greater than $-1$,

$$\zeta^* = M + (M^2 - 1)^{1/2} \tag{72}$$

and this equation necessarily gives a value for $\zeta^*$ such that $-1 \leq \zeta^* \leq 1$ and therefore $\zeta_0 = \pm\zeta^*$ according as $K' \gtrless 0$; there is now a surface separating loading and unloading regions and it has the equation $\zeta = \pm \zeta^*$ according as $K' \gtrless 0$, i.e. a surface composed of parts of the planes $\zeta = \pm \zeta^*$ with discontinuities at points such that $K' = 0$. Such discontinuities may be due to the approximate nature of the solution. Further $d = d^*(\pm\zeta_0)$ according as $K' \gtrless 0$ and therefore $d = d^*(\zeta^*)$ in both cases. Equation (58) is now

$$D_{11}w'_{,\xi\xi\xi\xi} + 2D_{12}w'_{,\xi\xi\eta\eta} + D_{22}w'_{,\eta\eta\eta\eta} = -(\sigma_1'w'_{,\xi\xi} + \sigma_2'w'_{,\eta\eta}) \tag{73}$$

where $D_{11}$, $D_{22}$ and $D_{12}$ (see Eqs. (4)) are constants dependent upon the stress-state. The problem therefore depends upon the solution of the linear, fourth order, homogeneous, partial differential equation (73) with constant coefficients dependent upon the stress-state, subject to boundary conditions on $w'$, e.g. Eqs. (69); the characteristic-values and characteristic-functions of this equation correspond respectively to the instability stresses and normal displacement-rates. Note that, although Eq. (73) has a doubly-infinite set of characteristic-values (and corresponding characteristic-functions), only the first is of importance from a practical standpoint, and further only in this case may Eq. (70) be a useful approximation to Eq. (65). Equation (73) is given in Ref. 2 and also, for the particular case $\sigma_2$ equal to zero, in Ref. 2. As remarked in Ref. 2, there is no serious mathematical difficulty involved in the determination of the (approximate) instability stresses for the edge conditions that are of practical importance; indeed the only difficulties are in the computation, and it is useful to note that it is simplest to regard $\sigma_1$, $\sigma_2$ and $\alpha$ (or $\beta$) as known and to solve for $\beta$ (or $\alpha$).

*Problem B.* Here it is postulated that the changes of the applied edge stresses at instability are such that plastic deformation only occurs.

It is evident on physical grounds that, if it is possible for loading, simulateously with bending, to occur over the whole of the plate, then the plate bending stiffness is least and therefore so also are the instability stresses. An analysis based on this assumption, in any case, must lead necessarily to a lower bound for instability stresses; the assumption is discussed later.

Then, by definition of the problem, $\zeta_0 = \mp 1$ according as $K' \gtrless 0$; notice, however, that the (plate) surface $\zeta = \pm 1$, in general, is not a boundary to a loading or an unloading region. Further $d = d^*(-1) = 1$ in both cases. Equation (58) is now

$$D_{11}w'_{,\xi\xi\xi\xi} + 2D_{12}w'_{,\xi\xi\eta\eta} + D_{22}w'_{,\eta\eta\eta\eta} = -(\sigma_1'w'_{,\xi\xi} + \sigma_2'w'_{,\eta\eta}) \tag{74}$$

where $D_{11}$, $D_{22}$ and $D_{12}$ (given from Eqs. (4) after setting $d = 1$) are constants dependent upon the stress state. Equations (73) and (74) are identical in form, and therefore the determination of the instability stresses in this problem is a task exactly comparable with that of the determination of the instability stresses in the previous problem. Equation (74) is given in Ref. 2 and also, for the particular case $\sigma_2$ equal to zero, in Ref. 3.

In the discussion of the assumption that plastic deformation only occurs, the analysis proceeds differently according as $K' > 0$ (Case $A$) or $K' < 0$ (Case $B$).

*Case A.* From Eq. (65) it follows that $\varphi'$ satisfies the equation

$$\nabla_1^4\varphi' = \beta\{\delta_1 r_1^{-1}w'_{,\xi\xi\xi\xi} - (\delta_1 + \delta_2)w'_{,\xi\xi\eta\eta} + \delta_2 r_2^{-1}w'_{,\eta\eta\eta\eta}\} \tag{75}$$

and satisfies boundary conditions necessarily consistent with the assumption that loading occurs everywhere. If this assumption is true, then at every point of the middle surface, it is untrue that $-1 \leq \zeta^*(P) \leq 1$. The critical case occurs when $\zeta^*(P) = -1$. Now suppose that, in general, at some particular point of the middle surface $\zeta^*(P) =$

$-(1 + \delta)$; for simplicity, it is assumed that $\delta$ is real. Then, from Eq. (45), it is straightforward to prove that

$$N\delta^2 - \delta + \delta_1 = 0 \tag{76}$$

where

$$\delta_1 = \{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(\beta \frac{K'}{L}\right) - 1 \tag{77}$$

and hence $\delta \geq \delta_1$ as $N \geq 0$. The critical case will occur if $\delta = 0$ and then $\delta_1 = 0$. A sufficient condition for loading to occur over the entire thickness of the plate is that $\delta_1 \geq 0$, i.e. from Eq. (77) that

$$\{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(\beta \frac{K'}{L}\right) \geq 1. \tag{78}$$

*Case B*. It is straightforward to show that the analysis of Case $B$ is deducible from the analysis of Case $A$. Full details are given in Ref. 4, and here only the results are stated. The relation corresponding to relation (78) is

$$\{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(-\beta \frac{K'}{L}\right) \geq 1. \tag{79}$$

Therefore a sufficient condition, applicable in either case, for loading to occur over the entire thickness of the plate is that

$$\{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(\frac{\beta}{L} \mid K' \mid\right) \geq 1. \tag{80}$$

Here it is required that loading occurs over the whole of the plate. Therefore it sufficient that $\varphi'$ is determined from Eq. (75) subject to boundary conditions consistent with the condition (80) for loading everywhere. The accurate determination of $\varphi'$ may not always be very simple. However, the following approximate analysis indicates that it should always be possible to choose the rate-of-change of applied edge stress at instability so that loading occurs over the entire plate, although the determination of the least possible rates of change is, in general, a difficult problem. Now if the instability stresses do not greatly exceed the elastic limit, then it is not unreasonable to approximate Eq. (75) by the equation

$$\nabla_1^4 \varphi' = 0. \tag{81}$$

Next take the boundary conditions on $\varphi'$ to be that (i) $\varphi'_{,ss}$ is constant along any edge and has the same value for opposite edges and (ii) $\varphi'_{,ns}$ is zero along all edges. The solution of Eq. (81) subject to these boundary conditions corresponds to a uniform stress-rate distribution. The condition (80) is now simply

$$\text{const.} \Big/ \frac{\beta}{L} K' \geq 1, \tag{82}$$

and loading will occur over the entire plate thickness if $\sigma_1' = -\varphi'_{,\eta\eta}$ and $\sigma_2' = -\varphi'_{,\xi\xi}$ are such that

$$(2\sigma_1 - \sigma_2)\sigma_1' + (2\sigma_2 - \sigma_1)\sigma_2' \geq \frac{\beta}{L} \text{Max.} \mid K' \mid. \tag{83}$$

The above analysis of Problems $A$ and $B$ refers to two salient problems of the general analysis; the first corresponds to greatest possible instability stresses and the second corresponds to least possible instability stresses; and, between these extremes, it appears probable that there exists a continuous set of instability stresses, this corresponding to a continuous set of changes of the applied edge stresses.

In Refs. 1, 2 and 3 some examples of Problems $A$, $A$ and $B$, and $B$, respectively are considered, but in all cases there is implicit neglect of strain compatibility. In Ref. 4, a detailed analysis is given of two cases, viz. (i) an infinitely-long simply-supported plate under uniform compression stress along one pair of edges and (ii) a finite simply-supported plate under uniform compression stress along one pair of opposite edges. Problems $A$ and $B$ are solved first accurately and then approximately (through neglect of strain compatibility) in case (i) and are solved approximately in case (ii). The approximate solutions may be expected to approximate closely the accurate solutions when the instability stresses are not too greatly in excess of the elastic limit.

**5. Conclusions.** There are two salient problems, often designated in the published literature as the von Kármán and Shanley problems; in the first, which corresponds to greatest possible instability stresses, it is postulated that the changes in the applied edge stresses at instability are zero; and in the second, which corresponds to least possible instability stresses, it is postulated that the changes in the applied edge stresses at instability are such that plastic deformation only occurs. It appears probable that, between these extremes, there exists a continuous set of instability stresses, this corresponding to a continuous set of changes in the applied edge stresses.

In the first problem, if strain compatibility is neglected, then the determination of the instability stresses is straightforward; it is probable that these approximate values underestimate the accurate values on the basis of the present theory by an amount that is small only if the instability stresses do not greatly exceed the elastic limit. In the second problem, the determination of the instability stresses is straightforward; however, although the assumption that only plastic deformation occurs at instability is probably correct, the determination of the corresponding least possible changes in the applied edge stresses is in general difficult.

It is known that theorectical values, based on the present flow theory, considerably overestimate experimental values of plastic buckling loads for long simply-supported plates under end compression. Further, deformation theories are available that provide results in good agreement with the experimental results; however, this fact is sometimes taken as a justification for a deformation, as opposed to a flow, theory of plasticity, an such a justification is untenable. If the experimental results relate to perfect test specimens perfectly loaded or, if these results are not particularly sensitive to imperfections of geometry, loading or mechanical behaviour, then the disagreement should be traceable to the assumed mechanical behaviour; otherwise, the disagreement should be traced to either imperfections or assumed mechanical behaviour or both of these matters.

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