—NOTES—

LARGE DEFLECTIONS OF A CANTILEVER BEAM WITH UNIFORMLY DISTRIBUTED LOAD*

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In this note only beams of uniform cross section are considered. The basic assumptions are that the deformations are elastic and that the bending does not alter the length of the beam.

Barten [1, 2] and Bisshopp and Drucker [3] have obtained results for a concentrated load. In the case of the uniformly distributed load, the Bernoulli-Euler equation,

$$\frac{1}{\rho} = \frac{d\theta}{ds} = \frac{M}{EI},$$

gives, with the origin at the free end of the beam,

$$\frac{dM}{ds} = -ws \cos \theta = EI \frac{d^2\theta}{ds^2}.$$

This equation does not lend itself to any simple solution. Put

$$\theta = \sum_{n=0}^{\infty} a_n s^n; \qquad \frac{d\theta}{ds} = \sum_{n=1}^{\infty} n a_n s^{n-1}; \qquad \frac{d^2\theta}{ds^2} = \sum_{n=2}^{\infty} n(n-1) a_n s^{n-2}.$$

The boundary conditions $s = 0$, $\theta = \alpha$, $d\theta/ds = 0$ give $a_0 = \alpha$; $a_1 = 0$. Then

$$\frac{d^2\theta}{ds^2} = -\frac{ws}{EI} (\cos \alpha \cos T - \sin \alpha \sin T), \qquad T = \sum_{n=2}^{\infty} a_n s^n.$$

Expanding $\cos T$ and $\sin T$ and comparing coefficients gives

$$a_{3n+4} = a_{3n+5} = 0, \qquad n = 0, 1, 2, \cdots; \qquad a_2 = 0;$$

$$a_3 = -w \cos \alpha / 6EI; \qquad a_6 = a_3 w \sin \alpha / 30EI;$$

etc. Thus

$$\theta = \alpha + \sum_{k=1}^{\infty} a_{3k} s^{3k}.$$

Since $dy/ds = \sin \theta$,

$$y = \int_0^s \sin \theta \, ds = T_1 \sin \alpha + T_2 \cos \alpha; \tag{1}$$

$$T_1 = s - a_3^2 s^7/14 - a_3 a_6 s^{10}/10 - \cdots;$$

$$T_2 = a_3 s^4/4 + a_6 s^7/7 + (a_9 - a_3^3/6) s^{10}/10 + \cdots.$$

Similarly,

$$x = T_1 \cos \alpha - T_2 \sin \alpha. \tag{2}$$

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Finally,

$$M = EI \sum_{k=1}^{\infty} 3ka_{3k}s^{3k-1};$$

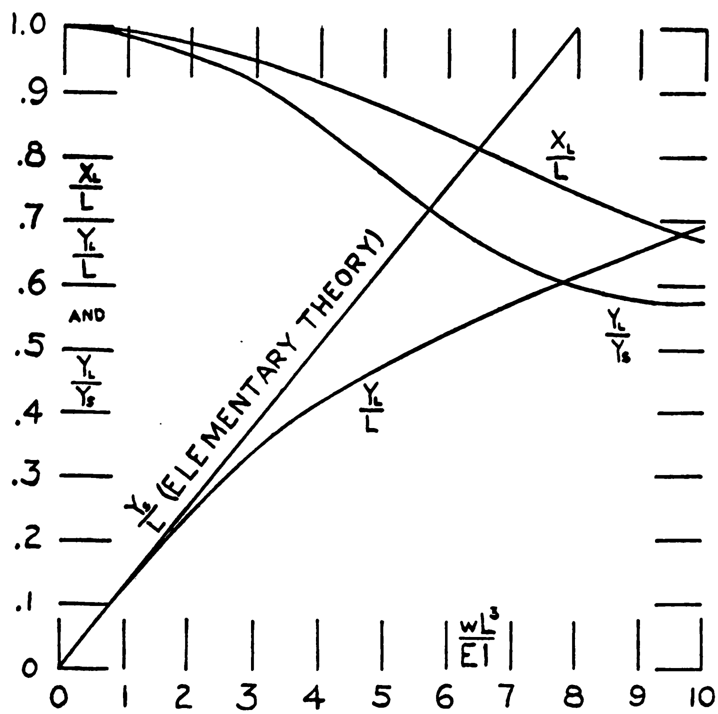
$$\alpha = -\sum_{k=1}^{\infty} a_{3k}L^{3k}. \tag{3}$$

With $s = L$, $\alpha = wL^3/6EI$, the first approximation to the maximum deflection is

$$y \doteq s \sin\alpha + \frac{1}{4} a_3 s^4 \cos\alpha \doteq wL^4/8EI,$$

which is in agreement with small deflection theory.

For computation, various values of α are chosen arbitrarily and corresponding values of wL^3/EI are found by eqn. (3). Against these values are plotted the ratios x_L/L, the horizontal projection of the deflection curve to the total length; y_L/L, the maximum deflection to the total length; and y_L/y_s, the maximum deflection to the maximum deflection as found by elementary theory. The results are shown in the figure.



For $wL^3/EI < 2$, values obtained by small deflection theory agree quite well with those obtained by large deflection theory. It is to be noted that the use of small deflection theory amounts to increasing the factor of safety.

BIBLIOGRAPHY

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