Adding to the expression (10) the contributions made by the loads OAE, OCG and OCF, and further, adding to the expression (11) the influence of loads OBG, ODE and ODF, the term A_{mn} for the whole plate is

$$A_{mn} = \frac{-16P(1 - \mu^2)}{\pi^6 EImn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{-2} \cos (m\pi/2) \cos (n\pi/2). \tag{12}$$

The expression (11) gives a trivial value of A_{mn} , because with both m and n odd, A_{mn} would vanish, which of course is impossible. It is seen that the equations (10) and (11) are true only for $m \neq n$, which, however, does not give any practical result. Moreover, we cannot set $m = n$, in the expression (10) and (11) because a value A_{mn} equal to infinity would result. It is clear that the integration performed is true only when m equals n; hence we must go back to the expression (9) and set there $m = n$. With this substitution expression (9) will yield the following value

$$\frac{1}{2\pi^2} \int_0^{\pi/2} u \sin 2mu \, du = \frac{1}{4m\pi}. \tag{13}$$

For the load OAH, the term A_{mn} from (8) becomes

$$A_{mn} = \frac{2P(1 - \mu^2)}{\pi^6 m^6 EI} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^{-2} \tag{14}$$

and for the whole plate

$$A_{mn} = \frac{16P(1 - \mu^2)}{\pi^6 m^6 EI} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^{-2}. \tag{15}$$

Finally the expression for the deflection is

$$w = \frac{16P(1 - \mu^2)}{\pi^6 EI} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^{-2} \left\{ \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{1}{3^6} \sin \frac{3\pi x}{a} \sin \frac{3\pi y}{b} + \cdots \right\}. \tag{16}$$

Thus, through this operation a double series for the deflection of the rectangular plate under pyramidal load is reduced to a result involving but a single series.

A RANDOM WALK RELATED TO THE CAPACITANCE OF THE CIRCULAR PLATE CONDENSER*

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Abstract. It is shown that the solution of Love's equation for the capacitance of the circular plate condenser can be expressed in terms of the mean duration of a certain one-dimensional random walk with absorbing barriers. The interpretation as a random walk makes it possible to confirm the fact that the actual capacitance of the condenser is always larger than the value given by the standard approximation for small separations, and yields an upper bound as well. In addition to its theoretical interest, the

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random walk appears to provide a practical means for the calculation of the capacitance by a Monte Carlo technique.

1. Introduction. The purpose of this note is to point out an equivalence between the mean number of steps of a random walk, and the capacitance C of a condenser consisting of two equal, infinitely thin circular conducting disks of radius a, separated by a distance κa.

It is a classical result that for $\kappa \to 0$, $C \sim a/4\kappa$, and it is also known (Jeans [1]) that $\lim_{\kappa\to\infty} C = a/\pi$. A more precise formula for small κ dates back to Kirchhoff [3], who also discusses a paper by Clausius on this subject. A formula of Nicholson's [5] said to give C for all κ by means of a definite integral has turned out to be incorrect. (Love [4] has pointed out fallacies in Nicholson's reasoning. Moreover, it can be shown by direct consideration of Nicholson's integral that it does not even give the correct asymptotic formula as $\kappa \to 0$). Present-day knowledge regarding the value of C for general values of κ appears to be restricted to the following elegant result obtained by Love.

LEMMA 1. *If $f(t)$ is the solution of the integral equation*

$$f(x) - \frac{1}{\pi}\int_{-1/\kappa}^{1/\kappa} \frac{f(t)}{1 + (x - t)^2}\, dt = 1 \qquad (-1/\kappa \le x \le 1/\kappa) \tag{1}$$

then

$$C = \frac{a\mu}{\pi}, \tag{2}$$

where

$$\mu = \frac{\kappa}{2}\int_{-1/\kappa}^{1/\kappa} f(t)\, dt. \tag{3}$$

It will be shown that both $f(t)$ and μ have very simple interpretations in terms of a random walk in the interval $|x| < 1/\kappa$. The results are expressed by the following theorem.

THEOREM 1. Let $\theta_i (i = 1, 2, \cdots)$ be independent random variables, such that $0 \le \theta_i \le 2\pi$, and Prob $\{\theta_i < A\} = 1/2\pi\, A$, $(0 \le A \le 2\pi)$. Consider the random walk starting at $x \,\varepsilon\, [-1/\kappa, 1/\kappa]$ whose i-th step equals $\tan \theta_i$. We will say that absorption occurs at step k if the k-th step is the first step landing outside $[-1/\kappa, 1/\kappa]$. *Then $f(x)$ is the expected number of steps to absorption.*

COROLLARY. Let the walk start at a random point in $[-1/\kappa, 1/\kappa]$ (i.e., a point drawn from a distribution with flat density). *Then μ is the expected number of steps to absorption.*

2. Derivation. It will be seen that Theorem 1 is a corollary of the following result:

THEOREM 2*. If

(a) $K(x,t)$ is continuous and ≥ 0 $(a \le x \le b,\ a \le t \le b)$,
(b) there exists a constant $\alpha > 0$ such that

$$\int_a^b K(x,t)\, dt \le 1 - \alpha \qquad \text{for all } x \varepsilon [a,b], \tag{4}$$

*See Wasow [6] for a different proof of this result under a less restrictive hypothesis.

then there exists a random walk starting at x whose mean number of steps to absorption is $f(x)$, where

$$f(x) - \int_a^b K(x,t)f(t)\,dt = 1 \qquad (a \le x \le b). \tag{5}$$

The dispersion of the number of steps will be finite.

First we derive the rather obvious

LEMMA 2. If ξ and η are random variables, taking on non-negative integral values, and

$$P_k = \text{Prob}\,\{\xi \ge k\} \le Q_k = \text{Prob}\,\{\eta \ge k\} \qquad (k = 0, 1, 2, \cdots)$$

then

$$\langle \xi \rangle \le \langle \eta \rangle \qquad \text{and} \qquad \langle \xi^2 \rangle \le \langle \eta^2 \rangle.$$

Proof:

$$\langle \eta \rangle = \sum_0^\infty k(Q_k - Q_{k+1}) \ge \sum_0^N k(Q_k - Q_{k+1}) + (N+1)Q_{N+1} = \sum_1^{N+1} Q_k\,.$$

Letting $N \to \infty$, we see that the term to the right of the inequality approaches $\langle \eta \rangle$, showing that $\langle \eta \rangle = \sum_1^\infty Q_k$, and consequently $\langle \xi \rangle \le \langle \eta \rangle$. The inequality for the second moments can be proved in the same way.

Proof of Theorem 2. For each $x \in [a,b]$ construct a continuous function $S(x,t)$ such that

(a) $S(x,t) \ge 0 \qquad (-\infty < t < \infty)$

(b) $\int_{-\infty}^{\infty} S(x,t)\,dt = 1$

(c) $S(x,t) = K(x,t) \qquad \text{for} \qquad a \le t \le b.$

For fixed x, the function $S(x,t)$ should be thought of as a probability density. The random walk is defined inductively as follows: We start at x, and stop when first landing outside $[a,b]$. Having taken n steps, and assuming the nth step lands us at $y \in [a, b]$, the next step is taken to the point λ, where

$$\text{Prob}\,\{\lambda < u\} = \int_{-\infty}^{u} S(y,t)\,dt.$$

Let $\xi = \xi(x) =$ number of steps to absorption if the random walk starts at x, and $P_k = P_k(x) = \text{Prob}\,\{\xi \ge k\}$. If before a given step occurs, absorption has not yet taken place, then the probability of absorption as a result of the step is not smaller than α. Thus $P_1 = 1$, and $P_k \le Q_k$, where $Q_k = (1-\alpha)^{k-1}$. By Lemma 2, $\langle \xi \rangle = f(x) \le 1/\alpha$, and $\langle \xi^2 \rangle \le (2-\alpha)/\alpha^2$. Dispersion $[\xi] = \langle (\xi - \langle \xi \rangle)^2 \rangle = \langle \xi^2 \rangle - \langle \xi \rangle^2 \le (2-\alpha)/\alpha^2$. Thus $f(x)$ exists, and has a finite dispersion.

Now to show that $f(x)$ satisfies (5).

Define $f(x) = 0$ for $x \notin [a,b]$. Then for all random walks whose first step carries to

y, the mean number of steps to absorption is

$$f(x \mid y) = 1 + f(y)$$

Therefore

$$\int_{-\infty}^{\infty} f(x \mid y) S(x,y)\, dy = \int_{-\infty}^{\infty} S(x,y)\, dy + \int_{-\infty}^{\infty} f(y) S(x,y)\, dy,$$

which reduces to

$$f(x) = 1 + \int_{a}^{b} f(y) K(x,y)\, dy$$

as was to be proved.

Theorem 2 follows from the observation that the tangent of a random number uniformly distributed in $(0,2\pi)$ has the probability density $(1/\pi)/(1 + x^2)$. The α of Theorem 2 can be taken as

$$1 - 1/\pi \int_{-1/\kappa}^{1/\kappa} \frac{dx}{1 + x^2} = (2/\pi) \text{ Arctan } \kappa.$$

Thus

$$f(x) \leq \frac{\pi}{2 \text{ Arctan } \kappa}, \qquad \text{and} \qquad C \leq \frac{a}{2 \text{ Arctan } \kappa} \qquad \text{for any } \kappa.$$

3. Further remarks. It is interesting to note that checks on the asymptotic values of C as $\kappa \to \infty$, and $\kappa \to 0$ are easily possible.

For $\kappa \to \infty$, we have of course $f(x) \to 1$, as can be seen either from the random walk, or the integral equation (1). Therefore $\mu \to 1$, and $C \to a/\pi$.

For $\kappa \to 0$ the formula $C \sim a/4\kappa$ can be obtained by using a result of Kac and Pollard [2] which also yields the conclusion that $C > a/4\kappa$. Kac and Pollard consider a *continuous* random motion on the x-axis, where if the particle is at point x_0 at time t_0, its position at time $t + t_0$ is given by a Cauchy distribution with semi-interquartile range t. We now make the observation that if one observes such a continuous random motion at the discrete time points $t = 0, 1, 2, \cdots$, and records the observed positions at these instants, the result is the generation of a discrete walk with exactly the properties defined in the hypothesis of Theorem 1.

Let $f^*(x)$ be the mean time to absorption of the *continuous* random motion, if it starts at x. As a simple consequence of the above observation we have $f(x) > f^*(x)$. Therefore

$$\mu > \kappa/2 \int_{-1/\kappa}^{1/\kappa} f^*(x)\, dx.$$

Using the result of Kac and Pollard for $E\{T(a,b,t)\}$ ([2], page 383) we obtain

$$f^*(x) = \left(\frac{1}{\kappa^2} - x^2\right)^{1/2}.$$

Therefore

$$C = \frac{a\mu}{\pi} > \frac{a\kappa}{2\pi} \int_{-1/\kappa}^{1/\kappa} f^*(x)\, dx = \frac{a}{4\kappa}. \tag{6}$$

In other words, C is always larger than the value predicted by the usual asymptotic formula for small separations. For $\kappa \to 0$ it is heuristically evident that $f(x) \sim f^*(x)$, so that $C \sim a/(4\kappa)$, as expected.†

A simple, plausible, "physical" proof of the inequality $C > a/4\kappa$ can be given.* C equals the ratio of charge per plate to potential difference between plates. The capacitor can be considered as being cut out of a pair of infinite parallel plates. Before the cut-away portions are removed the ratio of charge to potential difference is $a/4\kappa$. When the cut-away portions are removed each particle of charge moves away from the axis, and therefore the field intensity at each point of the axis is decreased below its previous value. Since the potential difference is the integral of this field intensity along the axis, the argument shows that the potential difference is reduced, and the capacitance therefore increased from $a/4\kappa$.

4. Practicability as a Monte Carlo method. If $\sigma = \langle(\xi - \mu)^2\rangle$ is the standard deviation of ξ, and n walks are taken, yielding $\xi_1, \xi_2, \cdots \xi_n$ as the observed values of ξ, then $\sigma/n^{1/2}$ will be the standard deviation of the empirical mean, and $\epsilon = \sigma/\mu n^{1/2}$ will be approximately equal to the ratio of standard deviation to the mean.

We will restrict ourselves to small κ, as it is conjectured that the n required for achieving a given ϵ steadily decreases as κ increases. For small κ, $\sigma < 2^{1/2}/\alpha \sim \pi/(2^{1/2}\kappa)$, $\mu > \pi/4\kappa$. Therefore, asymptotically for small κ,

$$\epsilon \leq (8/n)^{1/2}. \tag{7}$$

A numerical experiment, using $\kappa = 0.1$, and $n = 1000$ was made, using automatic computing machinery. The results were an empirical mean value of 10.101, and an ϵ of approximately 0.034. For $n = 1000$ the right-hand side of (7) yields $\epsilon \leq 0.09$.

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†*Note added in proof:* For a stronger result see, G. Polya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, 1951.

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