# APPROXIMATE SOLUTION OF AN INITIAL VALUE PROBLEM BY GENERALIZED CARDINAL SERIES*

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**1. Introduction.** A problem which arises in many different contexts is to approximate one of a certain class of solutions, $u(x,y)$, of a partial differential equation or integral equation by a function which interpolates its values on a line. More generally, let $V$ be a vector space, whose elements are functions of a point $P$ in a space $X$, so that the sum of two functions in $V$ belongs to $V$, as does the product of a function in $V$ by a constant. Consider the problem of obtaining a function of $V$ assuming given values $a_k$ at points $P_k$, for $k$ belonging to a set, $I$, of integers. If a family $\{A_k(P)\}$ of functions of $V$ can be determined so that $A_k(P_k) = 1$, $A_k(P_j) = 0$ for $j \neq k$ $(j,k \in I)$, then such an interpolatory function is given formally by the sum $\sum_{k \in I} a_k A_k(P)$. Such functions $A_k(P)$ can sometimes be defined as follows. Let, for each point $P$, $\varphi(P)$ denote an element in a Hilbert space $H$. Let $(\alpha,\beta)$ denote the inner product of two elements, $\alpha$ and $\beta$, of $H$, and let there exist elements $v_k$ of $H$ such that $(\varphi(P_k),v_k) = 1$, $(\varphi(P_k),v_j) = 0$, $j \neq k$, $j,k \in I$. Then one may put $A_k(P) = (\varphi(P),v_k)$, if this inner product belongs to $V$. Suppose, for example, that as functions of $t$, $\varphi(t;P)$ and $v_k(t)$ belong to the class $L_2$ (have integrable square) on an interval $(a,b)$ $(-\infty \leq a < b \leq \infty)$, that the integral $\int_a^b \varphi(t;P_j)v_k(t)\,dt$ is 1 for $j = k$ and 0 for $j \neq k$, $j,k \in I$, and that the integral belongs to $V$. Then one may put $A_k(P) = \int_a^b \varphi(t;P)v_k(t)dt$. If, as functions of $t$, the functions $\{\varphi(t;P_k)\}$ form an orthonormal system over $(a,b)$, one may set $v_k(t) = \varphi(t;P_k)$.

We shall consider only the case where $P$ represents a point in the $xy$-plane of a set $E$ containing the $x$-axis, where $v_k(t) \equiv \exp(ikt)$, and where $\phi(t;x,y)$ is defined for $(x,y) \in E$ so that

$$\varphi(t;\, x,\, 0) = \exp(-ixt). \tag{1.1}$$

We define

$$A_{k,\lambda}(x,\, y) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t/\lambda;\, x,\, y) \exp(ikt)\, dt \tag{1.2}$$

for each integer $k$, each positive number $\lambda$, and each point $(x,y)$ in $E$. The series

$$\sum_{k=-\infty}^{\infty} f(k\lambda) A_{k,\lambda}(x,y) \tag{1.3}$$

becomes, for $y = 0$,

$$\sum_{k=-\infty}^{\infty} f(k\lambda) \sin\left[\frac{\pi}{\lambda}(x - k\lambda)\right] \Big/ \frac{\pi}{\lambda}(x - k\lambda), \tag{1.4}$$

which is the cardinal series associated with values $f(k\lambda)$ at points $x = k\lambda$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. If a function $U(x,y)$ is given for $y = 0$, $U(x,0) \equiv f(x)$, the series (1.3) gives, formally, a function defined on $E$ coinciding with $f(x)$ at points $x = k\lambda$

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($k = \cdots, -2, -1, 0, 1, 2, \cdots$) on the $x$-axis. The advantages of the cardinal series as an interpolatory function are well-known. The cardinal series (1.4), under suitable conditions, yields a "smooth" function of $x$, in the sense that its Fourier transform vanishes outside the interval $(-\pi/\lambda, \pi/\lambda)$ ([1]; cf. also [2]). There is also a "consistency" property: a new cardinal series associated with values of the cardinal series (1.4) at equally-spaced points having adjacent points not farther than $\lambda$ units apart coincides with the cardinal series (1.4) ([3], [4], also [2]; for further references to the literature on cardinal series see [2], also [5]). In 1908 de la Vallée Poussin proved approximation theorems [6] giving conditions sufficient in order that the series (1.4) will approach $f(x)$ as $\lambda \to 0$. It is our purpose here also to obtain an approximation theorem: to obtain conditions sufficient in order that the series (1.3) should converge, and should approach $U(x,y)$ as $\lambda \to 0$, where $U(x,y)$ is a function of $V$, defined on $E$, uniquely determined by its values $f(x)$ on the $x$-axis. In §3, the method and the theorem of §2 are applied to the problem of approximating the temperature at a certain instant in an infinite insulated rod in terms of its temperatures at a later instant. As is well known, this problem has also a probabilistic interpretation. If a random variable **z** having a known frequency function is known to be the sum of independent random variables, **x** and **y**, the random variable **y** having a normal frequency function with mean 0 and standard deviation $\sigma$. §3 applies, and yields a method of numerical approximation to the frequency function of **x**. In order for the method to apply formally, it is not necessary that **y** be normally distributed.

**2. An approximation theorem.** In this section standard techniques of Fourier analysis are used to prove an approximation theorem, Theorem 2.1.

We observe that the function $\sin[\pi(x - k\lambda)/\lambda]/[\pi(x - k\lambda)/\lambda]$, and the function which vanishes outside $[-\pi/\lambda, \pi/\lambda]$ and which is given by $\lambda \exp(ik\lambda t)$ on $[-\pi/\lambda, \pi/\lambda]$, are a Fourier transform pair; i.e.,

$$\frac{\sin \pi(x - k\lambda)/\lambda}{\pi(x - k\lambda)/\lambda} \equiv \frac{\lambda}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} \exp(ik\lambda t) \exp(-ixt)\, dt. \tag{2.1}$$

If the series

$$\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp(ik\lambda t) \tag{2.2}$$

converges uniformly, then it is clear from (2.1) that the series (1.4) converges (throughout this paper, integrals and series extending from $-\infty$ to $\infty$ are said to converge if the Cauchy principal value exists, and converge uniformly if $\sum_{-p}^{p}$ or $\int_{-p}^{p}$ converges uniformly as $p \to \infty$). It is desirable, therefore, to determine conditions on $f(x)$ implying that the series (2.2) converges uniformly. It is simpler to state such conditions as conditions on the transform, $F(t)$, of $f(x)$. We have the following lemma.

Lemma 2.1: *If*

$$F(t) \text{ is continuous and of bounded variation on } (-\infty, \infty), \tag{2.3}$$

*if, for each positive number* $\lambda$,

$\sum_{j=-\infty}^{\infty} F(t + 2\pi j/\lambda)$ *converges uniformly and absolutely for* $|t| \leq \pi/\lambda$ *to a function* $F_\lambda(t)$, (2.4) *then the integral* $1/2\pi \int_{-\infty}^{\infty} F(t) \exp(-ixt)dt$ *converges to a function* $f(x)$, *and the series*

$$\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp(ik\lambda t) \tag{2.2}$$

*converges uniformly to* $F_\lambda(t)$; *moreover, for* $|t| \leq \pi/\lambda$,

$$\lim F_\lambda(t) = F(t), \tag{2.5}$$

*as* $1/\lambda \to \infty$ *through integral values.*

*Proof*: We have

$$\lambda \int_{-(2p+1)\pi/\lambda}^{(2p+1)\pi/\lambda} F(t) \exp(-ik\lambda t)\, dt = \lambda \sum_{j=-p}^{p} \int_{-\pi/\lambda}^{\pi/\lambda} F(t + 2\pi j/\lambda) \exp(-ik\lambda t)\, dt$$

$$= \int_{-\pi}^{\pi} \sum_{j=-p}^{p} F(u/\lambda + 2\pi j/\lambda) \exp(-iku)\, du.$$

By hypothesis, $\sum_{j=-\infty}^{\infty} F(u/\lambda + 2\pi j/\lambda)$ converges uniformly and absolutely to $F_\lambda(u/\lambda)$ for $|u| \leq \pi$. Since $\lambda$ is arbitrary (positive), it follows that $(1/2\pi) \int_{-\infty}^{\infty} F(t) \exp(-ixt)dt$ converges to a function $f(x)$, and that $\lambda f(k\lambda) = (1/2\pi) \int_{-\pi}^{\pi} F_\lambda(u/\lambda) \exp(-iku)du$. We observe that the total variation of $F_\lambda(u/\lambda)$ on the interval $|u| \leq \pi$ is not greater than the total variation of $F(t)$ on $(-\infty, \infty)$. Moreover, $F_\lambda(u/\lambda)$ is continuous (and of period $2\pi$, as a function of $u$). Hence its Fourier series, $\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp(iku)$, converges uniformly ([7], p. 42) to $F_\lambda(u/\lambda)$, so that $\lambda \sum_{k=-\infty}^{\infty} f(k\lambda) \exp(ik\lambda t)$ converges uniformly to $F_\lambda(t)$. Hypothesis (2.4) clearly implies that, for $|t| \leq \pi/\lambda$, $\lim_{\lambda \to 0} F_\lambda(t) = F(t)$; for let $t$ be fixed. We have $|F_\lambda(t) - F(t)| \leq \sum_{|j|\geq 1} |F(t + 2\pi j/\lambda)| \leq \sum_{|j|>N} |F(t + 2\pi j)|$ if $\lambda < 1/N$. But this latter sum is arbitrarily small for sufficiently large $N$. The proof of the lemma is complete.

The hypotheses on $F(t)$ of continuity and finite total variation may be replaced by others which imply the uniform convergence on $|u| \leq \pi$ of the Fourier series of $F_\lambda(u/\lambda)$. This lemma is essentially equivalent to Poisson's summation formula ([8], p. 33, ff.).

We recall that

$$\varphi(t; x, 0) = \exp(-ixt), \tag{1.1}$$

and

$$A_{k,\lambda}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t/\lambda; x, y) \exp(ikt)\, dt. \tag{1.2}$$

Using lemma 2.1, we obtain the following theorem.

THEOREM 2.1: *Let* $F(t)$ *satisfy the following conditions*:

$$F(t) \textit{ is continuous and of bounded variation on } (-\infty, \infty); \tag{2.3}$$

$\sum_{j=-\infty}^{\infty} F(t + 2\pi j/\lambda)$ converges uniformly and absolutely for

$$|t| \leq \pi/\lambda \textit{ to a function } F_\lambda(t); \tag{2.4}$$

*to each point* $(x,y)$ *in* $E$ *corresponds a function of* $t$, $L(t;x,y)$, *integrable with respect to* $t$ *over* $(-\infty, \infty)$, *such that, for each* $\lambda > 0$,

$$|F_\lambda(t)\varphi(t;x,y)| \leq L(t;x,y), \textit{ for } |t| \leq \pi/\lambda. \tag{2.6}$$

*Then the integral* $(1/2\pi)\int_{-\infty}^{\infty} F(t)\exp(-ixt)dt$ *converges to a function* $f(x)$. *The series* $\sum_k f(k\lambda)A_{k,\lambda}(x,y)$ *converges to the function* $f_\lambda(x,y) \equiv (1/2\pi)\int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t;x,y)dt$ *for* $(x,y) \in E$; $f_\lambda(x,0)$ *is the cardinal series associated with values* $f(k\lambda)$, *so that in particular,* $f_\lambda(k\lambda,0) = f(k\lambda)$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. *Moreover, as* $1/\lambda \to \infty$ *through integral values,* $f_\lambda(x,y) \to f(x,y)$ *for* $(x,y)$ *in* $E$, *where* $f(x,y) = (1/2\pi)\int_{-\infty}^{\infty} F(t)\varphi(t;x,y)\,dt$, *a function coinciding with* $f(x)$ *on the x-axis.**

*Proof*: That the integral $(1/2\pi)\int_{-\infty}^{\infty} F(t)\exp(-ixt)dt$ converges, is guaranteed by lemma 2.1. Also, by lemma 2.1, the series $\lambda\sum_{k=-\infty}^{\infty} f(k\lambda)\exp(ik\lambda t)$ converges uniformly for $|t| \leq \pi/\lambda$ to $F_\lambda(t)$. We have

$$\sum_{k=-n}^{n} f(k\lambda)A_{k,\lambda}(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} f(k\lambda)\exp(ikt)\varphi(t/\lambda; x, y)\,dt,$$

$$= \frac{\lambda}{2\pi}\int_{-\pi/\lambda}^{\pi/\lambda}\sum_{k=-n}^{n} f(k\lambda)\exp(ik\lambda u)\varphi(u; x, y)\,du.$$

Hence

$$\sum_{k=-n}^{n} f(k\lambda)A_{k,\lambda}(x, y) \qquad \text{converges to} \qquad f_\lambda(x, y), \tag{2.7}$$

where

$$f_\lambda(x, y) = \frac{1}{2\pi}\int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; x, y)\,dt. \tag{2.8}$$

By (1.4), $f_\lambda(x,0)$ is the cardinal series assuming values $f(k\lambda)$ at points $x = k\lambda$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. By (2.5), hypothesis (2.6), and Lebesgue's bounded convergence theorem, we have

$$\lim_{\lambda\to 0} f_\lambda(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\varphi(t; x, y)\,dt = f(x, y),$$

which completes the proof of the theorem.

Essentially, Theorem 2.1 gives conditions sufficient in order that an integral of the form $\int_{-\infty}^{\infty} F(t)\varphi(t)dt = \int_{-\infty}^{\infty}\varphi(t)dt\int_{-\infty}^{\infty} f(x)\exp(ixt)dx$ can be approximated by replacing $f(x)$ by the cardinal series associated with its values at points $x = k\lambda$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$, $\lambda > 0$. The parameters $(x,y)$ are mentioned explicitly in the above formulation with a view to the application in which it is desired to continue into a set $E$ of the $xy$-plane a function given on the $x$-axis, which belongs to a certain class.

**3. Temperature on an infinite insulated rod.** Let $a$ be a positive constant, and let $U(x,y)$ denote the temperature at the point with coordinate $x$ on an infinite, insulated rod, at time $y \geq -a$. It is known that if $U(x, -a)$ is piecewise continuous, bounded,

**Note added in proof:* The hypothesis in Theorems 2.1 and 3.1 that $F(t)$ is continuous and of bounded variation on $(-\infty, \infty)$ serves only to justify the term-by-term integration of the product of $\varphi(t/\lambda; x, y)$ by the Fourier series of $F_\lambda(t)$. Accordingly, hypothesis (2.3) in Theorem 2.1 may be replaced by the weaker hypothesis that $F_\lambda(t)$ is integrable over $[-\pi/\lambda, \pi/\lambda]$, if the further hypothesis that $\varphi(t/\lambda; x, y]$ is of bounded variation as a function of $t$ $(x, y, \lambda$ being fixed) on $[-\pi, \pi]$ is added. Since these hypotheses are satisfied in the special case to which Theorem 3.1 applies, in that theorem the hypotheses of continuity and bounded variation on $F(t)$ may be omitted. Correspondingly, in Corollary 3.1 the hypothesis that $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$ for some $\alpha > 1$ may be replaced by the hypothesis that $f(x)$ is absolutely integrable over $(-\infty, \infty)$.

and satisfies a Lipschitz condition, then the function $U(x,y)$ given by

$$U(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} U(\xi, -a) \exp \{-(\xi - x)^2/4(y + a)\} d\xi$$

is the unique solution of the heat equation for $y > -a$ which, together with its partial derivative with respect to $x$ is bounded ($|U(x,y)| \leq M, -\infty < x < \infty, y > -a$, $|U_x(x,y)| \leq M_2, |x| \geq x_1, y > -a$) and which approaches $U(x_0, -a)$ at a point of continuity as $(x,y)$ approaches $(x_0, -a)$ from above ($y > -a$).

Let $V$ be the class of functions $u(x,y)$, defined for $y \geq -a$, such that

$$u(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} u(\xi, -a) \exp \{-(\xi - x)^2 /4(y + a)\} d\xi \tag{3.1}$$

for $y > -a$. We take for the set $E$ the set of all points $(x,y)$ with $y > -a$. Let us suppose that $U(x,y) \varepsilon V$, that $U(x,0) \equiv f(x)$ is known, that $U(x, -a)$ is absolutely integrable over $(-\infty, \infty)$, and that it is desired to express $U(x,y)$ for $y > -a$ in terms of $f(x)$. We observe that for fixed $y > -a$, $U(x,y)$ is the convolution of the functions $U(x, -a)$ and $\exp \{-x^2/4(y + a)\}/2[\pi(y + a)]^{1/2}$. Thus if $V(t,y) = \int_{-\infty}^{\infty} U(x,y) \exp (ixt)dx$, we have

$$V(t, y) \equiv V(t, -a) \exp \{-t^2(y + a)\} \qquad \text{for} \qquad y > -a. \tag{3.2}$$

Hence

$$V(t, 0) = V(t, -a) \exp (-at^2), \tag{3.3}$$

and

$$V(t, y) = V(t, 0) \exp (-yt^2) \qquad \text{for} \qquad y > -a. \tag{3.4}$$

Since $U(x,y) = (1/2\pi) \int_{-\infty}^{\infty} V(t,y) \exp (-itx)dt$, we have

$$U(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp (-itx - yt^2) \, dt \int_{-\infty}^{\infty} U(\xi, 0) \exp (i\xi t) \, d\xi \tag{3.5}$$

for $y > -a$. This formula gives $U(x,y)$ in terms of $U(x,0) = f(x)$, but in a form unsuitable for numerical approximation. If $y > 0$, we may interchange the order of integration in (3.5), or use (3.4) directly, to obtain

$$U(x, y) = \frac{1}{2(\pi y)^{1/2}} \int_{-\infty}^{\infty} U(\xi, 0) \exp \{-(\xi - x)^2/4y\} \, d\xi, \tag{3.6}$$

but this formula is not available for $y < 0$. The method developed in §1 and §2 applies, however, and yields an interpolation formula,

$$f_\lambda(x, y) = \sum_{k=-\infty}^{\infty} f(k\lambda) A_{k,\lambda}(x, y), \tag{3.7}$$

which, under suitable conditions, approximates $U(x,y)$ for $y > -a$.

Let

$$\varphi(t; x, y) \equiv \exp (-itx - yt^2), \tag{3.8}$$

and

$$A_{k,\lambda}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t/\lambda; x, y) \exp (ikt) \, dt. \tag{1.2}$$

We have

$$A_{k,\lambda}(x, y) = \Phi(x/\lambda - k, y/\lambda^2), \tag{3.9}$$

where

$$\Phi(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-itx - yt^2)\, dt. \tag{3.10}$$

Applying Theorem 2.1, we obtain the following theorem.

THEOREM 3.1: *Let $U(x,y)$ be a function of the class $V$; i.e., let*

$$U(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} U(\xi, -a) \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi \quad for \quad y > -a.$$

*Let $U(x,-a)$ be absolutely integrable over $(-\infty, \infty)$. Let $F(t) \equiv V(t,0) \equiv \int_{-\infty}^{\infty} f(x) \exp(ixt)dt$, where $f(x) \equiv U(x,0)$, and suppose that $F(t)$ is continuous, and of bounded variation on $(-\infty, \infty)$. Then the series $\sum_{k=-\infty}^{\infty} f(k\lambda)\Phi(x/\lambda - k, y/\lambda^2)$ converges to a function $f_\lambda(x,y)$ of $V$ such that $f_\lambda(x,0)$ is the cardinal series associated with values $f(k\lambda)$, so that in particular $f_\lambda(k\lambda,0) = f(k\lambda)$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. Moreover, as $1/\lambda \to \infty$ through integral values, $f_\lambda(x,y)$ approaches $U(x,y)$ $(y > -a)$.*

Thus, if a sufficiently small unit interval is chosen on the $x$-axis, the formula $U(x,y) \simeq f_1(x,y) = \sum_{k=-\infty}^{\infty} f(k)\Phi(x - k,y)$ provides a method of numerical integration of (3.5), for negative as well as for positive values of $y$.

*Proof*: The above discussion, leading to equations (3.2) and (3.3), shows that $F(t) \equiv V(t,0)$ exists. Since $U(x,-a)$ is absolutely integrable over $(-\infty, \infty)$, the function $V(t,-a)$ is bounded. Equation (3.3) then insures that the series $\sum_{j=-\infty}^{\infty} F(t + 2\pi j/\lambda)$ converges unifirmly and absolutely for $|t| \leq \pi/\lambda$ to a function $F_\lambda(t)$ which is $0(\exp(-at^2))$. Hence, for $y > -a$, $F_\lambda(t)\varphi(t;x,y) = 0(\exp\{-(y + a)t^2\})$. Thus the hypotheses of Theorem 2.1 are satisfied, the set $E$ being the set of all points $(x,y)$ for which $y > -a$. We conclude that as $\lambda \to 0$, $f_\lambda(x,y) \to f(x,y) = (1/2\pi) \int_{-\infty}^{\infty} F(t)\varphi(t;x,y)dt$; but by (3.5) this is just $U(x,y)$. It remains to show that $f_\lambda(x,y) \in V$. One verifies easily that

$$\varphi(t; x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} \varphi(t; \xi, -a) \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi \tag{3.11}$$

for $y > -a$, i.e., $\phi(t;x,y) \in V$ for each $t$. By Theorem 2.1, and (3.11),

$$\begin{aligned} f_\lambda(x, y) &= \frac{1}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; x, y)\, dt \\ &= \frac{1}{4\pi^{3/2}(y + a)^{1/2}} \int_{-\pi/\lambda}^{\pi/\lambda} \exp(at^2)F_\lambda(t)\, dt \int_{-\infty}^{\infty} \exp\{-i\xi t - (\xi - x)^2/4(y + a)\}\, d\xi \\ &= \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi \left[\frac{1}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; \xi. -a)\, dt\right] \end{aligned}$$

for $y > -a$, the interchange of integrals being justified by virtue of the uniformity with respect to $t$ of the convergence of the integral

$$\int_{-\infty}^{\infty} \exp\{-i\xi t - (\xi - x)^2/4(y + a)\}\, d\xi.$$

But

$$f_\lambda(\xi, -a) = \frac{1}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} F_\lambda(t)\varphi(t; \xi, -a)\, dt,$$

hence

$$f_\lambda(x, y) = \frac{1}{2[\pi(y + a)]^{1/2}} \int_{-\infty}^{\infty} f_\lambda(\xi, -a) \exp\{-(\xi - x)^2/4(y + a)\}\, d\xi \qquad (y > -a),$$

i.e., $f_\lambda(x,y) \in V$. This completes the proof of the theorem.

It seems desirable to state conditions on $f$ alone, so far as is possible, sufficient for the conclusion of Theorem 3.1. To this end we prove the following lemma.

LEMMA 3.1: *If $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$, $\alpha \geq 0$, and if $\int_{-\infty}^{\infty} U((2a)^{1/2}v + x, -a) \exp(-v^2/2)dv$ converges uniformly with respect to $x$ (in particular if $U(x,-a)$ is bounded), then $x^\alpha U(x,-a)$ is absolutely integrable over $(-\infty, \infty)$.*

*Proof*: Suppose the contrary; then if $U_+$ and $U_-$ denote the positive and negative parts of $U(x,-a)$ respectively, either $x^\alpha U_+(x,-a)$ fails to be integrable over $(0,\infty)$, or over $(-\infty,0)$, or $x^\alpha U_-(x,-a)$ so fails. Suppose the first eventuality occurs. We have

$$f_+(x) = \frac{1}{2(\pi a)^{1/2}} \int_{-\infty}^{\infty} U_+(\xi, -a) \exp\{-(\xi - x)^2/4a\}\, d\xi$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} U_+([2a]^{1/2}v + x, -a) \exp(-v^2/2)\, dv,$$

$$\int_N^M |x|^\alpha f_+(x)\, dx = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp(-v^2/2)\, dv \int_{N+(2a)^{1/2}v}^{M+(2a)^{1/2}v} |t - (2a)^{1/2}v|^\alpha U_+(t, -a)\, dt$$

$$> \frac{1}{(2\pi)^{1/2}} \int_0^{1/(2a)^{1/2}} \exp(-v^2/2)\, dv \int_{N+1}^{M} (t - 1)^\alpha U_+(t, -a)\, dt$$

If $N > 0$. Now $(t - 1)^\alpha = t^\alpha(1 - 1/t)^\alpha \geq (1/2^\alpha)t^\alpha$ on $[N + 1, M]$ if $N > 1$, hence

$$\int_N^M x^\alpha f_+(x)\, dx > \frac{1}{2^\alpha} Erf\,[1/(2a)^{1/2}] \int_{N+1}^{M} t^\alpha U_+(t, -a)\, dt, \text{ where}$$

$Erf(z) = 1/(2\pi)^{1/2} \int_0^z \exp(-v^2/2)dv$. Since $\int_{N+1}^M t^\alpha U_+(t,-a)dt$ is arbitrarily large for proper choice of $N$, $M$, $x^\alpha f_+(x)$ is not integrable, contrary to the hypothesis. We obtain similarly a contradiction if $x^\alpha U_+$ is not integrable over $(-\infty,0)$, or if $x^\alpha U_-$ is not integrable over $(0,\infty)$, or over $(-\infty,0)$.

In connection with the hypotheses of Theorem 3.1, we now make the following observations:

(A) *If $\int_{-\infty}^{\infty} U[(2a)^{1/2}v + x,-a] \exp(-v^2/2)dv$ converges uniformly with respect to $x$ (in particular, if $U(x,-a)$ is bounded), and if $f(x) \equiv U(x,0)$ is absolutely integrable, so also is $U(x,-a)$.* This is an immediate consequence of Lemma 3.1, with $\alpha = 0$.

(B) *If $f(x)$ is absolutely integrable over $(-\infty, \infty)$, then $F(t) = \int_{-\infty}^{\infty} f(x) \exp(ixt)dx$ is continuous.* The proof is immediate.

(C) *If $U(x,-a) \sum_{n=-\infty}^{\infty} \exp\{-(an\pi/x)^2\}$ is absolutely integrable, then $F(t)$ is of bounded variation on $(-\infty, \infty)$.*

*Proof*: By (3.3), we have $F(t) = V(t,-a)\exp(-at^2)$, where $V(t,-a) = \int_{-\infty}^{\infty} U(x,-a)\exp(ixt)dx$. Hence

$$\sum_{i=M}^{N} | F_c(t_i'') - F_c(t_i') | = \sum_{i=M}^{N} | V_c(t_i'', -a)\exp(-at_i''^2) - V_c(t_i', -a)\exp(-at_i'^2) |,$$

where the subscript $c$ (for cosine transform) denotes the real part of the corresponding function. This latter sum is given by

$$\sum_{i=M}^{N} \left| \int_{-\infty}^{\infty} U(x, -a)[\exp(-at_i''^2)\cos xt_i'' - \exp(-at_i'^2)\cos xt_i']\, dx \right|,$$

which is not greater than

$$\int_{-\infty}^{\infty} | U(x, -a) | \sum_{i=M}^{N} | \exp(-at_i''^2)\cos xt_i'' - \exp(-at_i'^2)\cos xt_i' | \, dx.$$

Now the total variation of $\exp(-at^2)\cos xt$ on $(-\infty, \infty)$ is given by $2\sum_{n=-\infty}^{\infty}\exp\{-a(n\pi/x)^2\}$, hence the right hand member is not greater in absolute value than $K$, where $K = \int_{-\infty}^{\infty} | U(x,-a) | \sum_{n=-\infty}^{\infty}\exp\{-a(n\pi/x)^2\}dx$. Thus $F_c(t)$ is of bounded variation on $(-\infty, \infty)$. For $F_s(t)$ we replace $\cos xt$ by $\sin xt$. The total variation of $\exp(-at^2)\sin xt$ on $(-\infty, \infty)$ is given by $2\sum_{n=-\infty}^{\infty}\exp\{-a[(n+\frac{1}{2})\pi/x]^2\}$. But $\sum_{n=0}^{\infty}\exp\{-a[(n+\frac{1}{2})\pi/x]^2\} < \sum_{n=0}^{\infty}\exp\{-a(n\pi/x)^2\}$, and $\sum_{n=-\infty}^{-1}\exp\{-a[(n+\frac{1}{2})\pi/x]^2\} < \sum_{n=-\infty}^{0}\exp\{-a[(n+\frac{1}{2})\pi/x]^2\} = 1 + \sum_{n=0}^{\infty}\exp\{-a[(n+\frac{1}{2})\pi/x]^2\} < 1 + \sum_{n=0}^{\infty}\exp\{-a(n\pi/x)^2\}$. Hence $U(x,-a)\sum_{N=-\infty}^{\infty}\exp\{-a[(n+\frac{1}{2})\pi/x]^2\}$ is also absolutely integrable over $(-\infty, \infty)$, so that $F_s(t)$ is of bounded variation on $(-\infty, \infty)$ also.

(D) *If there exists a number $\alpha > 1$ such that $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$, and if $\int_{-\infty}^{\infty} U((2a)^{1/2}v + x, -a)\exp(-v^2/2)dv$ converges uniformly with respect to $x$ (in particular, if $U(x,-a)$ is bounded), then $F(t)$ is of bounded variation on $(-\infty, \infty)$.*

*Proof*: By lemma 3.1, $x^\alpha U(x,-a)$ is absolutely integrable over $(-\infty, \infty)$. But $\sum_{n=-\infty}^{\infty}\exp\{-a(n\pi/x)^2\} = o(x^\alpha)$ as $x \to \infty$, for $\alpha > 1$. To see this, we have only to observe that $\max_x x^{-b}\exp\{-a(n\pi/x)^2\} = \exp(-b/2)(b/2a\pi^2)^{b/2}(1/n)^b$. Hence $x^{-b}\sum_{n=-\infty}^{\infty}\exp\{-a(n\pi/x)^2\} < x^{-b} + 2\exp(-b/2)(b/2a\pi^2)^{b/2}\sum_{n=1}^{\infty}1/n^b$. On choosing $b$ between 1 and $\alpha$, it becomes clear that $x^{-\alpha}\sum_{n=-\infty}^{\infty}\exp\{-a(n\pi/x)^2\} \to 0$ as $x \to \infty$. Thus the hypotheses of (C) are satisfied.

These observations yield the following corrollary of Theorem 3.1:

Corollary 3.1: *If $U(x,y) \varepsilon V$, i.e., if*

$$U(x, y) = \frac{1}{2[\pi(y+a)]^{1/2}}\int_{-\infty}^{\infty} U(\xi, -a)\exp\{-(\xi - x)^2/4(y+a)\} \qquad for \qquad y > -a,$$

*if $U(x,-a)$ is bounded, and if $x^\alpha f(x)$ is absolutely integrable over $(-\infty, \infty)$ for some number $\alpha > 1$, where $f(x) \equiv U(x,0)$, then the series $\sum_{k=-\infty}^{\infty} f(k\lambda)\Phi(x/\lambda - k, y/\lambda^2)$ converges to a function $f_\lambda(x,y)$ of $V$ such that $f_\lambda(x,0)$ is the cardinal series associated with values $f(k\lambda)$ $(k = \cdots, -2, -1, 0, 1, 2, \cdots)$. Moreover, $f_\lambda(x,y) \to U(x,y)$ $(y > -a)$, as $1/\lambda \to \infty$ through integral values.*

**4. A probabilistic interpretation.** It is convenient to perform a translation of the $xy$-plane parallel to the $y$-axis, so that the point $(0,-y_0)$ $(0 < y_0 < a)$ becomes the new origin. Let $f^*(x)$ denote the function $U(x,-y_0)$, and $h^*(x)$ the function $U(x,0)$. Formula (3.4) yields

$$U(x,\,0) = \frac{1}{2(\pi y_0)^{1/2}} \int_{-\infty}^{\infty} U(\xi,\,-y_0) \exp\,\{-(\xi - x)^2/4y_0\}\,d\xi, \qquad \text{or}$$

$$h^*(x) = \frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{\infty} f^*(\xi) \exp\,\{-(\xi - x)^2/4y_0\}\,d\xi. \tag{4.1}$$

If $h^*$ is regarded as known, and $f^*$ as unknown, §3 gives a method of approximate solution of the integral equation (4.1).

Put $\sigma = (2y_0)^{1/2}$; we have

$$h^*(x) = \frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{\infty} f^*(\xi) \exp\,\{-(\xi - x)^2/2\sigma^2\}\,d\xi,$$

so that $h^*(x)$ is the expected value of a function $f^*(\mathbf{w})$ expressed in terms of the mean, $x$, of the normally distributed random variable $\mathbf{w}$ with standard deviation $\sigma$. Thus §3 provides a means of approximating a function $f(x)$, if the expected value of the function $f(\mathbf{w})$ is known in terms of the mean of the normally distributed random variable $\mathbf{w}$ having known standard deviation $\sigma$.

If $f(x)$ is a probability frequency function, then $h^*(x)$ is the probability frequency function of the sum of two independent random variables, one of which has the frequency function $f(x)$, while the other is normally distributed with mean 0 and standard deviation $\sigma$. Thus if it is known that a random variable $\mathbf{z}$ having a known distribution is the sum of two independent random variables, $\mathbf{x}$, and $\mathbf{y}$, $\mathbf{y}$ being normally distributed with mean 0 and standard deviation $\sigma$, the method of §3 can be used to approximate the frequency function of $\mathbf{x}$.

It is not necessary that $\mathbf{y}$ be normally distributed in order to apply the method formally. Let $\mathbf{x}$ have the unknown frequency function $f^*(x)$, $\mathbf{y}$ the known frequency function $g^*(y)$; let $\mathbf{x}$ and $\mathbf{y}$ be independent, and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$ have the known frequency function $h^*(z)$. Then $h^*(z) = \int_{-\infty}^{\infty} f^*(x)g^*(z - x)dx$. Let $H(t)$ denote the transform of $h^*$, $F(t)$ the transform of $f^*$, and $G(t)$ the transform of $g^*$. Formally, we have $H(t) = F(t)G(t)$, $F(t) = H(t)/G(t)$, and $f^*(x) = (1/2\pi) \int_{-\infty}^{\infty} \exp\,(-ixt)/G(t)dt \int_{-\infty}^{\infty} h^*(\xi) \exp\,(i\xi t)d\xi$. Now replace $h^*(\xi)$ by the cardinal series associated with its values at points $k\lambda (\lambda > 0,\, k$ integer):

$$h_\lambda(\xi) = \sum_{k=-\infty}^{\infty} h(k\lambda) \sin\frac{\pi}{\lambda}(x - k\lambda) \Big/ \frac{\pi}{\lambda}(x - k\lambda).$$

We obtain

$$f_\lambda(x) = \sum_{k=-\infty}^{\infty} h^*(k\lambda) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{G(t/\lambda)} \exp\,(-ixt/\lambda) \exp\,(ikt)\,dt. \tag{4.1}$$

Thus if we put $\varphi(t;x,y) = \exp\,(-ixt)$ for $y = 0$, $\varphi(t;x,y) = \exp\,(-ixt)/G(t)$ for $y = y_0$, and let $E$ be the set consisting of the lines $y = 0$, $y = y_0$, then equation (4.1) is the result of a formal application of the method to which Theorem 2.1 applies. However, it may be difficult, in individual cases, to verify the hypotheses of the theorem.

## References

1. E. T. Whittaker, *On the functions which are represented by the expansions of the interpolation-theory*, Proc. Roy. Soc. Edinburgh, **35**, 181-194 (1915).
2. J. M. Whittaker, *Interpolatory function theory*, Camb. Univ. Press, 1935.
3. W. L. Ferrar, *On the cardinal function of interpolation-theory*, Proc. Roy. Soc. Edinburgh, **45**, 269-282 (1925); **46**, 323-333 (1925).
4. W. L. Ferrar, *On the consistency of cardinal function interpolation*, Proc. Roy. Soc. Edinburgh, **49**, 230-242 (1927).
5. G. H. Hardy, *Notes on special systems of orthogonal functions*. IV. *The orthogonal functions of Whittaker's cardinal series*, Proc. Camb. Philos. Soc., **37**, 331-348 (1941).
6. C. -J. de la Vallée Poussin, *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, Bull. Acad. Roy. de Belg., Classe de Sci., 1, 319-410 (1908).
7. G. H. Hardy and W. W. Rogosinski, *Fourier series*, Camb. Univ. Press, 1950.
8. S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932.