THE TORSION AND STRETCHING OF SPIRAL RODS (II)*

BY H. ŌKUBO (*Tōhoku University, Sendai, Japan*)

The torsion and stretching problems of spiral rods were discussed in a preceding paper.[1] There, the equations of equilibrium were expressed in terms of displacements that were independent of the position of the section perpendicular to the axis of a spiral rod. The differential equations for the displacements were integrated for the particular case where the helix angle was small, and the corresponding displacements and stresses were obtained. In the calculations, however, the displacements were preliminarily assumed in special forms, and consequently solution was valid for some special problems. In the previous paper, the displacements for the stretching problem were assumed in forms that reduce to those for a uniform tension in the limit case when the helix angle approaches zero. But when a spiral rod with axis that does not pass the centroid of the cross section, is pulled axially, the displacements must be in forms that reduce to those for a uniform tension combined with a uniform bending moment in the limit case when the helix angle approaches zero. Hence, the validity of the previous solution was restricted to the problem for a spiral rod with axis through the centroid of the cross section.

As in the preceding paper, we take the axis of helix as the axis of z, and denote the displacements in x', y', z directions by u', v' and w respectively, in which x' and y' are the axes perpendicular to each other and fixed to a section of the rod perpendicular to z. We take for the displacements the expressions

$$\left.\begin{aligned} u' &= u_1 - \gamma x' - \frac{\gamma'}{2}(x'^2 - y'^2) - \alpha y'z + \frac{\beta'}{k^2}(1 - \cos kz - kz \sin kz), \\ v' &= v_1 - \gamma y' - \gamma' x'y' + \alpha x'z - \frac{\beta'}{k^2}(\sin kz - kz \cos kz), \\ w &= w_1 + \frac{\beta'}{k}(x' \sin kz - y' \cos kz + y') + \beta z, \end{aligned}\right\} \tag{1}$$

where u_1, v_1, w_1 are the functions of x', y' and are independent of z, k is a constant which specifies the helix angle, α, β, β' are arbitrary constants, and

$$\gamma = \frac{1}{2}(1 - p)\beta, \qquad \gamma' = \frac{1}{2}(1 - p)\beta', \qquad p = \frac{\mu}{\lambda + \mu}.$$

From (1) we have the cubical dilatation

$$\Delta = \frac{\partial u_1}{\partial x'} + \frac{\partial v_1}{\partial y'} - kD_2(w_1) + p\beta' x' + p\beta,$$

where

$$D_2 = y' \frac{\partial}{\partial x'} - x' \frac{\partial}{\partial y'}.$$

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[1] H. Ōkubo, Q. Appl. Math. 9, 263-272 (1951).

The equations of equilibrium for this case can be expressed in the forms

$$\left.\begin{aligned}\frac{\partial\Delta}{\partial x'} + p\{\nabla_1^2 u_1 + k^2 D_1(u_1) - 2k^2 D_2(v_1) + \frac{1}{2}k^2\gamma'(x'^2 - y'^2) - \beta'\} &= 0,\\ \frac{\partial\Delta}{\partial y'} + p\{\nabla_1^2 v_1 + k^2 D_1(v_1) + 2k^2 D_2(u_1) + k^2\gamma' x' y'\} &= 0,\\ kD_2(\Delta) - p\{\nabla_1^2 w_1 + k^2 D_1(w_1) + k^2 w_1 - k\beta' y'\} &= 0,\end{aligned}\right\} \tag{2}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}, \qquad \text{and} \qquad D_1 + 1 = \left(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}\right)\left(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}\right).$$

The differential equations (2) are independent of z. Solving the simultaneous equations for u_1, v_1 and w_1, we can find the displacements from (1). The displacements for a straight rod are readily obtained from (1), by taking the limit case when k approaches zero; thus

$$\left.\begin{aligned} u &= -\gamma x - \alpha yz - \frac{\gamma'}{2}(x^2 - y^2) - \frac{\beta'}{2}z^2,\\ v &= -\gamma y + \alpha xz - \gamma' xy,\\ w &= w_1 + \beta z + \beta' xz,\end{aligned}\right\} \tag{3}$$

where u_1, v_1 are assumed to vanish when k approaches zero.

Assume that w_1 and α in (3) also vanish when k approaches zero; the corresponding stresses become

$$X_x = Y_y = X_y = X_z = Y_z = 0, \qquad Z_z = \mu(3 - p)(\beta + \beta' x). \tag{4}$$

This is the solution for a straight rod submitted to simple tension combined with a uniform bending moment. If we assume that β and β', instead of w_1 and α, vanish when k approaches zero, the corresponding stresses become

$$X_x = Y_y = Z_z = X_y = 0, \qquad X_z = \mu\left(\frac{\partial w_1}{\partial x} - \alpha y\right), \qquad Y_z = \mu\left(\frac{\partial w_1}{\partial y} + \alpha x\right). \tag{5}$$

This is the solution for the torsion problem of a straight rod.

Consider now a spiral rod of small k, pulled by a pair of axial forces. Assuming that α, w_1 are small quantities of the order k and u_1, v_1 are of the order k^2, and neglecting the small quanties of the higher order, the equations of equilibrium (2) can be written as follows:

$$\left.\begin{aligned}\frac{\partial\Delta}{\partial x'} + p\nabla_1^2 u_1 + \frac{1}{2}pk^2\gamma'(x'^2 - y'^2) - p\beta' &= 0,\\ \frac{\partial\Delta}{\partial y'} + p\nabla_1^2 v_1 + pk^2\gamma' x' y' &= 0,\\ \nabla_1^2 w_1 - 2k\beta' y' &= 0.\end{aligned}\right\} \tag{6}$$

Let us take for w_1 the expression

$$w_1 = i(f_3 - \bar{f}_3) + \tfrac{1}{3}k\beta' y'^3, \tag{7}$$

where f_3 is an arbitrary function, $\bar{f}_3$ is a function conjugate with f_3, and

$$f_3 = f_3(\zeta), \qquad \bar{f}_3 = \bar{f}_3(\bar{\zeta}), \qquad \zeta = x' + iy', \qquad \bar{\zeta} = x' - iy'.$$

The displacement w_1 satisfies the third Eq. (6). Substituting this expression of w_1 into the first and second equations (6), the equations for u_1 and v_1 become

$$\left.\begin{aligned}
&(1 + p)\frac{\partial^2 u_1}{\partial x'^2} + \frac{\partial^2 v_1}{\partial x'\,\partial y'} + p\frac{\partial^2 u_1}{\partial y'^2}\\
&\qquad = k(f_3' + \bar{f}_3' + \zeta f_3'' + \overline{\zeta f_3''}) - \frac{1}{2}pk^2\gamma' x'^2 - \frac{1}{4}(4 - p + p^2)k^2\beta' y'^2,\\
&p\frac{\partial^2 v_1}{\partial x'^2} + \frac{\partial^2 u_1}{\partial x'\partial y'} + (1 + p)\frac{\partial^2 v_1}{\partial y'^2}\\
&\qquad = ik(f_3' - \bar{f}_3' + \zeta f_3'' - \overline{\zeta f_3''}) - \frac{1}{2}(4 + p - p^2)k^2\beta' x'y'.
\end{aligned}\right\} \tag{8}$$

Integrating the differential equations (8), we find

$$\left.\begin{aligned}
u_1 &= f_1 + \bar{f}_1 + x'(f_2 + \bar{f}_2) + k\int f_3'\zeta\,d\zeta + k\int \bar{f}_3'\bar{\zeta}\,d\bar{\zeta}\\
&\quad + \frac{k^2\beta'}{48}\{p(3 - p)x'^4 - 6(4 + p - p^2)x'^2y'^2 + (6 - p - p^2)y'^4\},\\
v_1 &= i(f_1 - \bar{f}_1) + ix'(f_2 - \bar{f}_2) + i(1 + 2p)\left(\int f_2\,d\zeta - \int \bar{f}_2\,d\bar{\zeta}\right),
\end{aligned}\right\} \tag{9}$$

where f_1, f_2 are arbitrary functions of ζ. The corresponding stresses become

$$\left.\begin{aligned}
X_{x'}' &= 2\mu\Big\{f_1' + \bar{f}_1' + x'(f_2' + \bar{f}_2') + p(f_2 + \bar{f}_2) + k(\zeta f_3' + \overline{\zeta f_3'})\\
&\qquad + \frac{1}{24}(3 - p)(1 + p)k^2\beta' x'^3 - \frac{1}{8}(9 - p^2)k^2\beta' x'y'^2\Big\},\\
Y_{y'}' &= -2\mu\Big\{f_1' + \bar{f}_1' + x'(f_2' + \bar{f}_2') + (2 + p)(f_2 + \bar{f}_2)\\
&\qquad - \frac{1}{24}(3 - p)(1 - p)k^2\beta' x'^3 + \frac{1}{8}(1 - p)^2k^2\beta' x'y'^2\Big\},\\
Z_z &= -2\mu\Big\{(1 - p)(f_2 + \bar{f}_2) + k(\zeta f_3' + \overline{\zeta f_3'}) - \frac{1}{24}(1 - p)(3 - p)k^2\beta' x'^3\\
&\qquad - \frac{1}{8}(7 + 2p - p^2)k^2\beta' x'y'^2 - \frac{1}{2}(3 - p)(\beta + \beta' x')\Big\};\\
X_{y'}' &= 2i\mu\Big\{f_1' - \bar{f}_1' + x'(f_2' - \bar{f}_2') + (1 + p)(f_2 - \bar{f}_2) + \frac{k}{2}(\zeta f_3' - \overline{\zeta f_3'})\\
&\qquad + \frac{i}{8}(4 + p - p^2)k^2\beta' x'^2y' - \frac{i}{24}(6 - p - p^2)k^2\beta' y'^3\Big\},\\
X_z' &= \mu\{i(f_3' - \bar{f}_3') - \alpha y' + k\gamma' x'y'\},\\
Y_z' &= -\mu\Big\{f_3' + \bar{f}_3' - \alpha x' - k\beta' y'^2 + \frac{k\gamma'}{2}(x'^2 - y'^2)\Big\}.
\end{aligned}\right\} \tag{10}$$

Take for the bounding curve of the section, the expression

$$F(x', y') = 0. \tag{11}$$

The conditions for the lateral surface of the rod to be free from external forces are

$$\left.\begin{aligned} X'_{x'}\frac{\partial F}{\partial x'} + X'_{y'}\frac{\partial F}{\partial y'} - kD_2(F)X'_z &= 0, \\ X'_{y'}\frac{\partial F}{\partial x'} + Y'_{y'}\frac{\partial F}{\partial y'} - kD_2(F)Y'_z &= 0, \\ X'_z\frac{\partial F}{\partial x'} + Y'_z\frac{\partial F}{\partial y'} - \mu k(3-p)(\beta+\beta' x')D_2(F) &= 0. \end{aligned}\right\} \tag{12}$$

Consider a spiral rod stretched by a pair of axial forces P, and imagine a small portion of the rod cut by two parallel planes perpendicular to the axis of rod, as shown in Fig. 1.

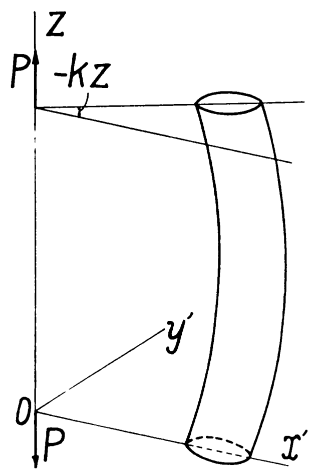

FIG. 1.

The equilibrium condition of the surface tractions for this portion is

$$\left.\begin{aligned} \iint Z_z\, dx'\, dy' &= P, \\ \iint Z_z x'\, dx'\, dy' &= 0, \\ \iint (y'X'_z - x'Y'_z)\, dx'\, dy' &= 0, \end{aligned}\right\} \tag{13}$$

where the integrals are taken over the cross section.

The arbitrary functions f_1 , f_2 and f_3 are determined so as to satisfy the boundary condition (12), and the constants α, β and β' are obtained from the condition (13).

Substituting the expressions of X'_z, Y'_z in (10) into the third Eq. (12), it becomes

$$\frac{d}{ds}(f_3 + \bar{f}_3) - \frac{1}{2}\{\alpha + (3 - p)k\beta\}\frac{d}{ds}(\zeta\bar{\zeta}) - \frac{1}{12}(11 - 3p)k\beta'\frac{dx'^3}{ds} - \frac{1}{4}(5 - p)k\beta'\frac{d}{ds}x'y'^2 = 0,$$

where ds is the element of arc of the bounding curve of the cross section. It follows that the equation

$$f_3 + \bar{f}_3 = \frac{1}{2}\{\alpha + (3 - p)k\beta\}(x'^2 + y'^2) + \frac{1}{12}(11 - 3p)k\beta' x'^3 + \frac{1}{4}(5 - p)k\beta' x'y'^2 + \text{const.}, \tag{14}$$

holds on the bounding curve, from which f_3 is determined. For the convenience of further calculations, we shall rewrite the stresses $X'_{x'}$, $Y'_{y'}$ and $X'_{y'}$ in (11) in the expressions as

$$\left.\begin{aligned} X'_{x'} &= \frac{\partial^2 \chi}{\partial y'^2} + 2\mu k(\zeta f'_3 + \bar{\zeta}\bar{f}'_3) + \frac{\mu}{12}(3 - p)(1 + p)k^2\beta' x'^3 - \frac{\mu}{4}(9 - p^2)k^2\beta' x'y'^2, \\ &\vdots \end{aligned}\right\} \tag{15}$$

where

$$\chi = -\varphi_1 - \bar{\varphi}_1 - x'(\varphi_2 + \bar{\varphi}_2), \qquad \varphi_1'' = 2\mu(f'_1 + pf_2), \qquad \varphi'_2 = 2\mu f_2 .$$

Substituting these expressions for stresses into the first and second Eqs. (12), by virtue of (14), we obtain

$$\left.\begin{aligned} &\frac{d}{ds}\left(\frac{\partial \chi}{\partial x'} + i\frac{\partial \chi}{\partial y'}\right) + \mu k\frac{d}{ds}[F_3(\zeta) - \bar{F}_3(\bar{\zeta})] + \mu k^2\beta'\frac{d\Phi}{ds} \\ &\qquad - \frac{1}{2}(3 - p)\mu k^2\beta'\zeta\frac{d}{ds}(\zeta\bar{\zeta}) - \frac{i}{4}(3 - p)\mu k^2\beta' x'^2 y'\frac{d}{ds}(\zeta + 3\bar{\zeta}) = 0, \\ &\frac{d}{ds}\left(\frac{\partial \chi}{\partial x'} - i\frac{\partial \chi}{\partial y'}\right) - \mu k\frac{d}{ds}[F_3(\zeta) - \bar{F}_3(\bar{\zeta})] + \mu k^2\beta'\frac{d\bar{\Phi}}{ds} \\ &\qquad - \frac{1}{2}(3 - p)\mu k^2\beta'\bar{\zeta}\frac{d}{ds}(\zeta\bar{\zeta}) + \frac{i}{4}(3 - p)\mu k^2\beta' x'^2 y'\frac{d}{ds}(3\zeta + \bar{\zeta}) = 0, \end{aligned}\right\} \tag{16}$$

where

$$F'_3 = 2\zeta f'_3 ,$$

$$\Phi = -\frac{1}{48}[(11 + p)(3 - p)x'^4 + 6(1 - p)^2x'^2y'^2 - (9 - 2p + p^2)y'^4]$$

$$+ \frac{i}{12}(3 - p)[(1 + p)x'^2 - (7 + p)y'^2]x'y'.$$

From (16), f_1 and f_2 are obtained.

As an example of the procedure, consider an elliptic spiral rod whose cross section is

$$(x' - c)^2/a^2 + y'^2/b^2 = 1, \tag{17}$$

as shown in Fig. 2.

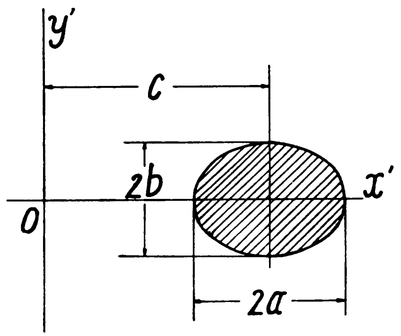

FIG. 2.

Transform now the elliptic section in the ζ-plane into a unit circle in the t-plane by the equation

$$\zeta = c + a't + b't^{-1}, \tag{18}$$

where $a' + b' = a$, $a' - b' = b$.

Take for the function f_3, the expression

$$f_3 = c_1(t + st^{-1}) + c_2(t^2 + s^2t^{-2}) + c_3(t^3 + s^3t^{-3}), \tag{19}$$

where $S = b'/a'$. The unknown coefficients c_1, c_2 and c_3 are readily obtained from the boundary condition (14).

Remembering the condition that φ_1 and φ_2 are analytic at any point of the section in the t-plane, we take for the functions the expressions

$$\varphi_1' = \sum_{n=1}^{4} A_n(t^n + s^nt^{-n}), \qquad \varphi_2 = \sum_{n=1}^{4} B_n(t^n + s^nt^{-n}), \tag{20}$$

where A_n and B_n are real constants. These unknown constants are obtained from the boundary condition (16), and the other unknown constants α, β and β' are finally obtained from the conditions (13).

In the case of stretching, the predominating stress is the normal stress Z_z and the shearing stresses X_z' and Y_z' follow it, but the latter are smaller quantities of the order k. The other stresses are of the order k^2 and are very small quantities when k is small. When the section of the spiral rod is a circle of radius unity, then $a = b = 1$, and it follows $a' = 1$, $b' = 0$, $s = 0$. The main stress Z_z for the circular section, referred to the polar coordinates with the pole at the center of the circle, is

$$
\begin{aligned}
Z_z = {} & \mu(3 - p)(\beta + \beta' c) - 2(1 - p)B_1 - 4\mu k c_1 c \\
& + [\mu\beta'(3 - p) - 4(1 - p)B_2 - 4\mu k(2c_2 c + c_1)]r \cos\theta \\
& - 2[3(1 - p)B_3 + 2\mu k(3c_3 c + 2c_2)]r^2 \cos 2\theta - 4[2(1 - p)B_4 + 3\mu k c_3]r^3 \cos 3\theta \\
& + \frac{1}{12}\mu k^2 \beta'(1 - p)(3 - p)(c + r\cos\theta)^3 \\
& + \frac{1}{4}\mu k^2 \beta'(7 + 2p - p^2)(c + r\cos\theta)r^2 \sin^2\theta.
\end{aligned}
\tag{21}
$$

From the third Eq. (13), we have

$$\alpha = \tfrac{1}{2}(1 - p)k\beta' c. \tag{22}$$

Hence, a twist almost proportional to k arises when a spiral rod with a circular section

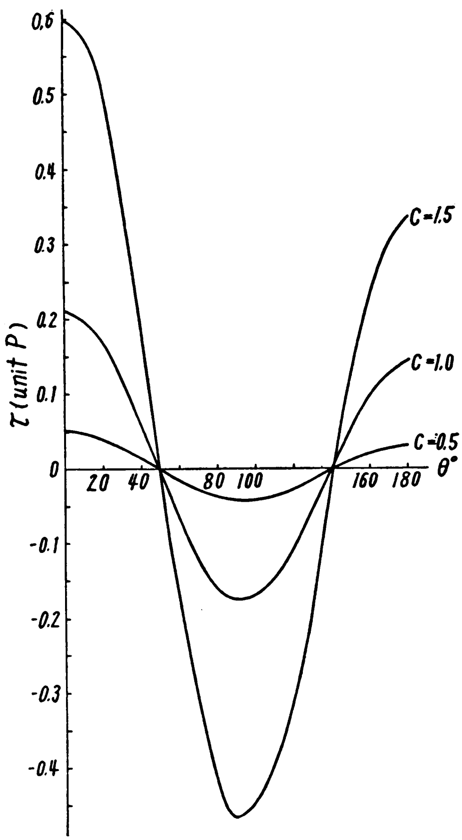

Fig. 3. The shearing stress along the bounding circle, when $k = 0.25$.

is pulled axially.[2] From the remaining Eqs. (13), the other unknown constants β and β' are obtained.

If τ be the shearing stress along the bounding circle of the section, it becomes

$$\tau = -\mu\{[2c_1 - \tfrac{1}{4}k\beta' c^2(1-p) - \tfrac{1}{4}pk\beta'] \cos\theta + 4c_2 \cos 2\theta + (6c_3 + \tfrac{1}{4}k\beta') \cos 3\theta\}. \quad (23)$$

The shearing stress along the periphery has been calculated from (23) for various values of c, assuming the Poisson's ratio and k to be 0.3 and 0.25, respectively, and is shown in Fig. 3. As is seen from the figure, the shearing stress becomes large at both ends of two diameters parallel to the coordinate axes (x', y'), and attains its maximum value at the outer end of the diameter on the x'-axis. The distribution of the normal tension Z_z on the axis of x', obtained from (21), is given in Fig. 4. For the sake of comparison, the

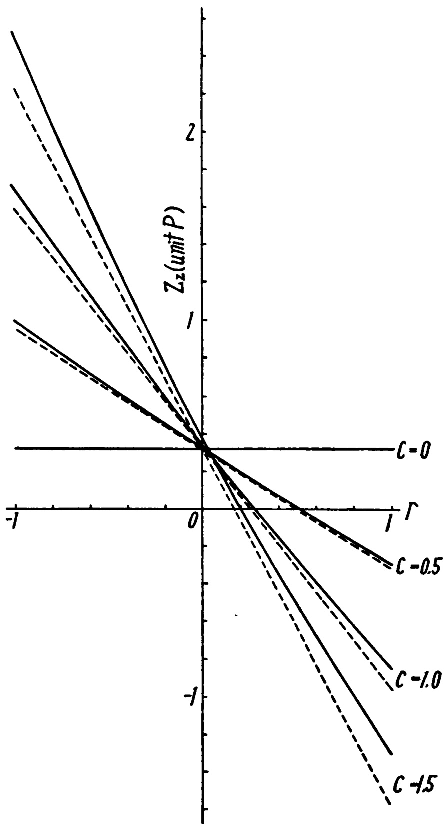

FIG. 4. The distribution of Z_z on the x'-axis, when $k = 0.25$.

corresponding distribution of Z_z for a straight rod is also shown by dotted lines in the same figure.

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[2] β' is a function of k and c, but it remains almost constant for the variation of k, when k is small.