Multiplying through by the denominator, we see that this derivative vanishes if and only if τ^2 satisfies

$$BG^2\tau^4 + (2BFG + EG)\tau^2 + (BF^2 - EF) = 0. \tag{9}$$

Our interest is confined to $M > 1$, $\gamma > 1$. Then it may easily be shown, using (8′), that the two roots of the equation are real; further, one of these roots is negative and has no physical significance since it leads to an imaginary value of τ.

A graph of $\lambda(M, \gamma)$, computed by using formula (9), is shown as Fig. 2.

Bibliography

[1] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience, 1948.

[2] H. W. Liepmann and A. E. Puckett, *Introduction to aerodynamics of a compressible fluid*, John Wiley and Sons, 1947, 51-60.

[3] J. M. Walsh, R. G. Shreffler and F. J. Willig, *Limiting Conditions for jet formation in high velocity collisions*, J. Appl. Phys. 24 (1953), 349-359.

ON THE THEORY OF THE BULGE TEST[1]

By E. W. ROSS, JR. and W. PRAGER (*Brown University*)

Summary. It is shown that the use of Tresca's yield condition and the associated low rule leads to a simple theory for the bulge test for perfectly plastic or strain-hardening naterials. The basic equations can be integrated in closed form even for finite deflections.

1. Introduction. The ductility of sheet metal under balanced biaxial tension is letermined by the bulge test: a circular sheet of uniform thickness is clamped round he periphery and subjected to unilateral fluid pressure which causes the sheet to bulge ılastically. The strain at the pole of the bulge is measured by means of a grid inscribed n the originally flat sheet, and the stress at the pole is computed from the applied ıressure and the measured curvature and thickness of the deformed sheet. The dimenions of the sheet are chosen so that its flexural stiffness is negligible[2]; on the other hand, he strains cannot be treated as infinitesimal.

The first consistent theory of the bulge test was given by Hill.[3] This theory is based n the yield condition and flow rule of v. Mises.[4] It is assumed that the relevant quantities ın be represented as power series in the ratio between the maximum deflection and the ıdius of the die aperture through which the sheet is made to bulge. Powers higher than

[1]Received Nov. 13, 1953. The results presented in the paper were obtained in the course of research ıonsored by Watertown Arsenal Laboratory under Contract DA-19-020-ORD-2598.

[2]Even for a very thin sheet neglecting the flexural stiffness may not be justified in the neighborhood the edge. Such edge effects are known to be highly localized, however, and may therefore be neglected the discussion of the states of stress and strain in the neighborhood of the pole of the bulge. For a scussion of plastic edge effects the reader is referred to a paper by F. K. G. Odqvist (Reissner Anniverry Volume, J. W. Edwards, Ann Arbor, Mich., 1949, p. 449).

[3]R. Hill, Phil. Mag. (7) 41, 1133-1142 (1950).

[4]R. v. Mises, Goettinger Nachrichten 1913, 582-592 (1913).

the second are neglected in Hill's analysis which is therefore restricted to moderate deflections.

The present paper contains an alternative theory of the bulge test. This theory is based on Tresca's yield condition[5] and the associated flow rule[6]; its equations can be solved in closed form without the use of special assumptions concerning the magnitude of the deflection.

2. Yield condition and flow rule. The middle surface of the bulged sheet is a surface of revolution, and the principal stresses at a generic point of this surface are directed along the parallel circle, the meridian, and the normal. In this order the principal stresses will be denoted by σ_c, σ_m, and σ_n; the initial thickness of the sheet will be denoted by h_0 and the radius of the die aperture by a. For the usual small values of the ratio h_0/a, the stress σ_n is much smaller than the stresses σ_c and σ_m. The state of stress at any point of the sheet is therefore essentially one of biaxial tension with the principal stresses σ_c and σ_m.

In the following, elastic strains will be neglected and the sheet material will be treated as incompressible. The principal plastic strain rates will be denoted by ϵ_c, ϵ_m, and ϵ_n, and the yield stress in simple tension by σ.

Tresca's yield condition specifies that, for plastic flow to occur, the maximum shearing stress must have an intensity, $\sigma/2$, that depends on the state of hardening of the material. If the three principal stresses are unequal, the state of flow is supposed to be one of pure shear in the plane determined by the largest and smallest principal stresses. For the states of biaxial tension considered here, the following basic possibilities must be considered:

a) If $\sigma_c > \sigma_m > 0$ during plastic flow, then

$$\sigma_c = \sigma \quad \text{and} \quad \epsilon_m = 0, \quad \epsilon_c = -\epsilon_n > 0;$$

b) if $\sigma_m > \sigma_c > 0$ during plastic flow, then

$$\sigma_m = \sigma \quad \text{and} \quad \epsilon_c = 0, \quad \epsilon_m = -\epsilon_n > 0.$$

Finally, it is assumed that any combination of the flow mechanisms specified under a) and b) is possible when $\sigma_c = \sigma_m$. Thus, a third case is added to the list:

c) if $\sigma_c = \sigma_m > 0$ during plastic flow, then

$$\sigma_c = \sigma_m = \sigma \quad \text{and} \quad \epsilon_c > 0, \quad \epsilon_m > 0, \quad \epsilon_n = -(\epsilon_c + \epsilon_m).$$

The flow rule c) is a natural extension of the flow rules a) and b); in fact, it represents the only way in which a continuous transition from a) to b) can be achieved.

For the purpose of comparison, we state the yield condition and flow rule of v. Mises for biaxial tension with the principal stresses σ_c, σ_m and the principal strain rates ϵ_c, ϵ_m:

$$\sigma_c^2 - \sigma_c\sigma_m + \sigma_m^2 = \sigma^2,$$

$$\frac{\epsilon_c}{\epsilon_m} = \frac{2\sigma_c - \sigma_m}{2\sigma_m - \sigma_c}, \qquad \text{sign } \epsilon_c = \text{sign } (2\sigma_c - \sigma_m).$$

---

[5]H. Tresca, Mémoires prés. par div. savants, 18, 733-799 (1868).

[6]W. Prager, *On the use of singular yield conditions and associated flow rules*, J. Appl. Mech. 20, 317-320 (1953).

This yield condition and flow rule has the great advantage of being valid throughout the entire range of biaxial tension, thus avoiding the necessity of discussing separate cases such as a), b), c) above. On the other hand, each of these three cases has the advantage that the values of two of the quantities σ_c , σ_m , ϵ_c , ϵ_m are known outright. This fact is responsible for the considerable mathematical simplifications obtained by the use of Tresca's yield condition and the associated flow rule which was first pointed out by Koiter.[7]

3. Kinematical considerations. It will be shown in Section 4 that the formation of a spherical bulge of uniform thickness is compatible with Tresca's yield condition and the associated flow rule. In this section, the kinematical relations will be derived which apply during the formation of such a bulge.

Figure 1 shows the circular meridian of the middle surface of the bulged sheet:

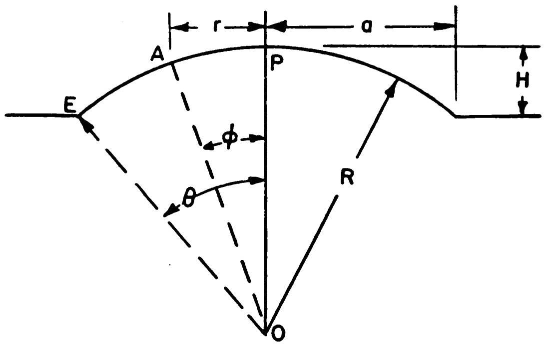

FIG. 1.

O is the center, R the radius, and A a generic point of the meridian; P is the pole, and E represents the edge. The angle POE will be denoted by θ and the angle POA by φ. Since the radius a of the die aperture is given and

$$R = a/\sin \theta, \tag{1}$$

the considered stage of the deformation is conveniently specified by the parameter θ. All "rates" used in the following are rates of change with respect to θ.

Denote by h the uniform sheet thickness in the bulged state and by h_0 that in the undeformed state. Since the sheet material is assumed to be incompressible, it must occupy equal volumina in both states. It follows from this condition that

$$h = h_0 \cos^2 \frac{\theta}{2} . \tag{2}$$

The "strain rate" ϵ_n is thus given by

$$\epsilon_n = \frac{1}{h} \frac{dh}{d\theta} = -\tan \frac{\theta}{2} . \tag{3}$$

[7]W. T. Koiter, C. B. Biezeno Anniversary Volume, H. Stam, Haarlem, Holland, 1953, p. 232.

Since the height of the bulge is

$$H = R(1 - \cos\theta) = a \tan\frac{\theta}{2}, \tag{4}$$

we have

$$\epsilon_n = -H/a. \tag{5}$$

Let r be the radius of the parallel circle through A and r_0 the radius of the corresponding circle in the undeformed sheet. When the condition of incompressibility is applied, not to the entire sheet but only to the portion which is bounded by the circle of radius r_0 in the undeformed state and by the circle of radius r in the deformed state, the following relation is obtained:

$$r = r_0\left(\frac{h_0}{h}\right)^{1/2}, \qquad \cos\frac{\varphi}{2} = r_0\frac{\cos\varphi/2}{\cos\theta/2}. \tag{6}$$

Now,

$$r = R\sin\varphi = a\frac{\sin\varphi}{\sin\theta}. \tag{7}$$

Eliminating r between (6) and (7) and solving for r_0 we obtain

$$r_0 = a\frac{\sin\varphi/2}{\sin\theta/2}. \tag{8}$$

For a given particle r_0 remains constant during the deformation. Differentiating (8) with respect to θ and using (6) and (8), we therefore obtain

$$\frac{d\varphi}{d\theta} = \frac{\tan\varphi/2}{\tan\theta/2} = \frac{r_0^2}{ar}. \tag{9}$$

From (6) and (9) it is seen that the circumferential strain rate is given by

$$\epsilon_c = \frac{1}{r}\frac{dr}{d\theta} = \frac{d}{d\theta}(\log r) = \frac{1}{2}\left[\tan\frac{\theta}{2} - \cot\frac{\theta}{2}\tan^2\frac{\varphi}{2}\right]. \tag{10}$$

From (3), (10), and the condition of incompressibility, the meridional strain rate is then obtained as

$$\epsilon_m = -\epsilon_c - \epsilon_n = \frac{1}{2}\left[\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\tan^2\frac{\varphi}{2}\right]. \tag{11}$$

The circumferential strain rate vanishes at the edge ($\varphi = \pm\theta$); for relevant values of θ, say $0 < \theta < \pi/2$, and all values of φ between $-\theta$ and θ, the strain rates (10) and (11) are seen to be positive. The state of flow considered here therefore is of the type c), and $\sigma_c = \sigma_m = \sigma$.

4. Static considerations. If $\sigma_c = \sigma_m = \sigma$, the equilibrium of the assumed spherical bulge of constant wall thickness under the applied pressure p requires that σ has the constant value

$$\sigma = \frac{pR}{2h} = \frac{pa}{2h_0\sin\theta\cos^2\theta/2} \tag{12}$$

in the entire bulge. This means that, at any given stage of the bulging, the entire material is in the same state of hardening.

Consider first a perfectly plastic material that flows under the constant stress σ_0 in simple tension. We then have $\sigma = \sigma_0 =$ const., and hence, from (12),

$$p = \frac{4\sigma_0 h_0}{a} \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2} . \tag{13}$$

As θ increases, the pressure (13) reaches the maximum value

$$\max p = \frac{3\sqrt{3}\, \sigma_0 h}{4a} \tag{14}$$

when $\theta = 60°$. The considered spherical bulge of constant wall thickness is not stable beyond this pressure maximum. When the pressure maximum is reached, $h/h_0 = 3/4$ according to (2). The logarithmic strain in the direction normal to the sheet therefore is nearly -30%.

Consider next a material that strain-hardens according to

$$\sigma = \sigma_0(1 + \alpha \mid \mathcal{E} \mid) \tag{15}$$

in simple tension, $\mathcal{E}$ being the logarithmic strain, and σ_0 and α being constants. The considered state of stress in the bulged sheet has the principal values $\sigma_c = \sigma_m = \sigma$, $\sigma_n = 0$; it may be obtained by the superposition of the state of balanced triaxial tension $\sigma_c = \sigma_m = \sigma_n = \sigma$ and the state of simple compression $\sigma_c = \sigma_m = 0$, $\sigma_n = -\sigma$. The first of these states of stress will not produce plastic flow or strain-hardening of the incompressible material, the second may be assumed to produce strain hardening in accordance with (15) provided that $\mathcal{E}$ in this formula is replaced by $\log (h/h_0)$, where h/h_0 is given by (2). Substituting

$$\sigma = \sigma_0\left(1 - 2\alpha \log \cos \frac{\theta}{2}\right) \tag{16}$$

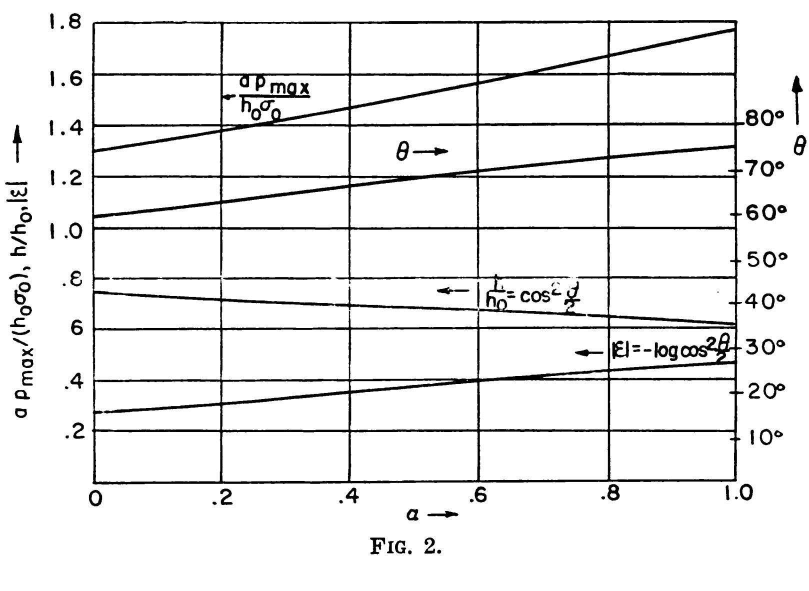

FIG. 2.

into (12) and solving for p, we obtain

$$p = \frac{4\sigma_0 h_0}{a}\left(1 - 2\alpha \log \cos \frac{\theta}{2}\right) \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2} . \tag{17}$$

Figure 2 shows the pressure maximum computed from (17) and the corresponding values of θ, h/h_0, and $|\,\varepsilon\,| = -\log (h/h_0)$, all versus the strain-hardening parameter α. It is seen that the pressure maximum $p_{\max}$ as well as the corresponding values of θ and $|\,\xi\,|$ increase with α, whereas the ratio h/h_0 at the pressure maximum decreases with α. In the considered range of α all these quantities vary with α in a nearly linear manner.

BOOK REVIEWS

Mathematical methods for scientists and engineers. By Lloyd P. Smith. Prentice-Hall, Inc., New York, 1953. x + 453 pp. $10.00.

The material in this book is essentially that taught for a number of years by the author to graduate students in physics and physical chemistry at Cornell University. The unusual feature of this text is the wide range of mathematical methods treated. At the same time it will be found that the treatment of the material is quite adequate for dealing with most physical problems. The discussion is consise and clear.

There is no specific treatment of differential equations except as they arise in the treatment of the other topics.

The reviewer believes that this is one of the most useful books available on intermediate and advanced mathematical methods.

The chapter headings of this text are as follows: Elements of Function Theory; Differential Calculus, Integral Calculus; Space Geometry; Line, Surface, and Multiple Integrals; Theory of Functions of a Complex Variable Residues and Complex Integration; Representation of Functions by Infinite Series of Functions; Applications of Functions of a Complex Variable to Potential and Flow Problems; Algebra of Linear Equations, Transformations and Quadratic Forms; Vector and Tensor Analysis; Orthonormal Function Systems; Orthonormal Functions with a Continuous Spectrum; Integral Equations; Variational Methods; and Elements of Probability Theory.

ROHN TRUELL

Calculus of variations with applications to physics and engineering. By Robert Weinstock. McGraw-Hill Book Company, Inc., New York, 1952. x + 326 pp. $6.50.

According to the preface, this volume presents an introduction to the calculus of variations followed by application of the subject to problems of physics and theoretical engineering.

The first five chapters give the usual elementary treatment of the calculus of variations with no pretense of complete mathematical rigor. Chapters 6-12 present applications to dynamics, elasticity, quantum mechanics, and electrostatics. These applications are, for the greater part, of an elementary nature; modern problems in acoustics, electromagnetic theory, and quantum mechanics are not discussed.

Up to the present time there is no other volume in the English language that offers such a variety of applications of the calculus of variations to problems in physics. The book must therefore be accepted as a worthwhile contribution to the applied mathematician's library.

This reviewer has a few adverse comments to make on the contents of the book; these are not in-