$y/(x^2 + y^2)] =$ constant along C. That is, the inversion of the boundary with respect to the circle must also be a streamline of the original flow $f(z)$. To be generally admissible this inversion of the boundary must be a streamline for every flow hence must itself be a rigid boundary. Thus a necessary and sufficient condition that rigid boundaries be generally admissible is that they be mapped into themselves under inversion with respect to the circle. A given boundary can be made conditionally admissible by obtaining the flow for this boundary together with its inversion and then regarding the inversion as only a streamline and not a boundary.

In particular, any plane barrier along $y = cx$, $a \le x \le b$, is generally admissible subject only to the condition that $ab = (1 + c^2)^{-1}$. As an example of the application of the circle theorem under this modified restriction on rigid boundaries, consider

$$f(z) = \frac{2}{3} \cos \theta[4z - 5] - i \frac{8}{3} \sin \theta\left[(z - 2)\left(z - \frac{1}{2}\right)\right]^{1/2}$$

representing flow at an angle θ about the plane boundary $1/2 \le x \le 2$. Then under the transformation $g(z) = f(z) + f^*(1/z)$ it can be shown that both the unit circle and the plane boundary are streamlines. Any portion of this plane boundary is conditionally admissible with respect to this complex potential $f(z)$.

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A NOTE ON A PAPER BY G. C. McVITTIE*

By G. B. WHITHAM (*The University of Manchester*)

In a recent paper** in the Quarterly of Applied Mathematics G. C. McVittie uses Einstein's equations in general relativity to derive certain solutions of the classical equations of continuity and momentum for the compressible flow of a fluid when heat conduction and viscosity are neglected. Explicit expressions (which always satisfy these equations) are obtained for the density ρ, pressure p and velocity components U_i in terms of arbitrary functions ϕ, ϕ_1, ϕ_2, ϕ_3 of the time T and space coordinates X_i; the ϕ's are not independent however and must satisfy certain (differential) consistency equations. The equations of continuity and momentum are (using the double suffix summation convention)

$$\frac{\partial \rho}{\partial T} + \frac{\partial}{\partial X_j}(\rho U_j) = 0, \tag{1}$$

$$\rho\left(\frac{\partial U_i}{\partial T} + U_j \frac{\partial U_i}{\partial X_j}\right) + \frac{\partial p}{\partial X_i} = 0; \tag{2}$$

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**G. C. McVittie, "*A method of solution of the equations of classical gas dynamics, using Einstein's equations*", Quart. of Appl. Math., 11, No. 3 (1953).

McVittie's solutions are

$$U_i = -\frac{\partial^2\phi}{\partial X_i\,\partial T}\Big/\nabla^2\phi, \tag{3}$$

$$\rho = -\nabla^2\phi, \tag{4}$$

$$p = -\frac{\partial^2\phi}{\partial T^2} + \frac{1}{3}\sum_{i=1}^{3}\left(\nabla^2\phi_i - \frac{\partial^2\phi_i}{\partial X_i^2}\right) + \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\partial^2\phi}{\partial X_i\,\partial T}\right)^2\Big/\nabla^2\phi, \tag{5}$$

where, if l, m, n is a cyclic permutation of 1, 2, 3,

$$\frac{\partial^2\phi_l}{\partial X_m\,\partial X_n} = \frac{\partial^2\phi}{\partial X_m\,\partial T}\,\frac{\partial^2\phi}{\partial X_n\,\partial T}\Big/\nabla^2\phi, \tag{6}$$

and

$$\begin{aligned}\frac{\partial^2\phi_2}{\partial X_3^2} + \frac{\partial^2\phi_3}{\partial X_2^2} + \left(\frac{\partial^2\phi}{\partial X_1\,\partial T}\right)^2\frac{1}{\nabla^2\phi} &= \frac{\partial^2\phi_1}{\partial X_3^2} + \frac{\partial^2\phi_3}{\partial X_1^2} + \left(\frac{\partial^2\phi}{\partial X_2\,\partial T}\right)^2\frac{1}{\nabla^2\phi}\\ &= \frac{\partial^2\phi_1}{\partial X_2^2} + \frac{\partial^2\phi_2}{\partial X_1^2} + \left(\frac{\partial^2\phi}{\partial X_3\,\partial T}\right)^2\frac{1}{\nabla^2\phi}.\end{aligned} \tag{7}$$

In any problem ϕ_1, ϕ_2, ϕ_3 must be eliminated from (6) and (7) so that p, ρ, U_i are expressed in terms of ϕ alone; usually ϕ would then have to satisfy some energy relation.

The purpose of this note is to point out two results which would be of considerable value to anyone using this solution: (i) the elimination of ϕ_1, ϕ_2, ϕ_3 can be carried out explicitly, and (ii) the results can be obtained in a few lines (and in fact more generally) directly from (1) and (2). This latter derivation is set out first.

It is clear that (1) can be satisfied by the introduction of an arbitrary vector A_i such that

$$\rho = \frac{\partial A_i}{\partial X_i}, \qquad \rho U_i = -\frac{\partial A_i}{\partial T}. \tag{8}$$

Equation (2) then expresses $\partial p/\partial X_i$ in terms of A_i. The simplest form is obtained if (1) is multiplied by U_i and added to (2) to give

$$-\frac{\partial p}{\partial X_i} = \frac{\partial}{\partial T}(\rho U_i) + \frac{\partial}{\partial X_j}(\rho U_i U_j), \tag{9}$$

whence, on substitution from (8),

$$-\frac{\partial p}{\partial X_i} = -\frac{\partial^2 A_i}{\partial T^2} + \frac{\partial}{\partial X_j}\left\{\frac{\partial A_i}{\partial T}\,\frac{\partial A_j}{\partial T}\Big/\frac{\partial A_k}{\partial X_k}\right\}. \tag{10}$$

Equations (8) and (10) give explicit expressions for ρ, U_i and p in terms of the arbitrary vector A_i. A special case may be obtained by choosing $A_i = -\partial\phi/\partial X_i$; this assumes [from (8)] that curl $\rho U_i = 0$, a condition which is always true if the flow depends only upon a *single* space variable and the time, but otherwise is restrictive. Then

$$U_i = -\frac{\partial^2\phi}{\partial X_i\,\partial T}\Big/\nabla^2\phi, \qquad \rho = -\nabla^2\phi, \qquad p = -\frac{\partial^2\phi}{\partial T^2} + \Phi, \tag{11}$$

where Φ is determined in terms of ϕ by

$$\frac{\partial \Phi}{\partial X_i} = \frac{\partial}{\partial X_j}\left\{\frac{\partial^2 \phi}{\partial X_i\, \partial T}\, \frac{\partial^2 \phi}{\partial X_j\, \partial T}\Big/ \nabla^2\phi\right\}. \tag{12}$$

Expressions (11) and (12) are McVittie's results *but with the dependence of p on ϕ given explicitly*. It is necessary to point out, however, that neither Eq. (10) nor Eq. (12) has a solution unless the right-hand side is an irrotational vector field. Since (12) is equivalent to McVittie's equations, this shows that these too have solutions only for a restricted class of ϕ.

It is now shown how (3), (4), (5), (6), (7) may be reduced to (11) and (12). To eliminate ϕ_1, ϕ_2, ϕ_3 in (5), (6), (7), an expression for the term $\frac{1}{3}\sum_{i=1}^{3}(\nabla^2\phi_i - \partial^2\phi_i/\partial X_i^2)$ in (5) is required in terms of ϕ. However, if the three quantities which are equated in (7) are denoted by E_1, E_2 and E_3, respectively, and $\frac{1}{3}(E_1 + E_2 + E_3)$ is denoted by Ψ, it it seen that (5) is $p = -\partial^2\phi/\partial T^2 + \Psi$; it remains, therefore, to demonstrate that $\Psi = \Phi$. Differentiating E_1 with respect to X_1 we have

$$\frac{\partial \Psi}{\partial X_1} = \frac{\partial E_1}{\partial X_1} = \frac{\partial^3 \phi_2}{\partial X_3^2\, \partial X_1} + \frac{\partial^2 \phi_3}{\partial X_2^2\, \partial X_1} + \frac{\partial}{\partial X_1}\left\{\left(\frac{\partial^2 \phi}{\partial X_1\, \partial T}\right)^2 \Big/ \nabla^2\phi\right\}. \tag{13}$$

Expressions for $\partial^2\phi_2/\partial X_3 \partial X_1$ and $\partial^2\phi_3/\partial X_2 \partial X_1$ are given in terms of ϕ by (6); hence $\partial\Psi/\partial X_1$ can be given in terms of ϕ alone. We have

$$\begin{aligned}\frac{\partial \Psi}{\partial X_1} &= \frac{\partial}{\partial X_3}\left\{\frac{\partial^2 \phi}{\partial X_3\, \partial T}\cdot\frac{\partial^2 \phi}{\partial X_1\, \partial T}\Big/ \nabla^2\phi\right\} + \frac{\partial}{\partial X_2}\left\{\frac{\partial^2 \phi}{\partial X_2\, \partial T}\cdot\frac{\partial^2 \phi}{\partial X_1\, \partial T}\Big/ \nabla^2\phi\right\} \\ &\qquad + \frac{\partial}{\partial X_1}\left\{\left(\frac{\partial^2 \phi}{\partial X_1\, \partial T}\right)\Big/ \nabla^2\phi\right\} \\ &= \frac{\partial}{\partial X_j}\left\{\frac{\partial^2 \phi}{\partial X_j\, \partial T}\cdot\frac{\partial^2 \phi}{\partial X_1\, \partial T}\Big/ \nabla^2\phi\right\}.\end{aligned} \tag{14}$$

$\partial\Psi/\partial X_2$ and $\partial\Psi/\partial X_3$ are obtained similarly and we observe that the expressions agree with (12); hence, Φ and Ψ are indeed the same.

BOOK REVIEWS

An introduction to the theory of differential equations. By Walter Leighton. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1952. viii + 174 pp. $3.50.

This book constitutes a carefully written exposition of the elements of the theory of ordinary linear differential equations. Accordingly, considerable attention has been given to questions of rigor, and to the understanding of basic concepts as opposed to mere formal facility of operation. To this end the author discusses the nature of solutions of differential equations and states and discusses existence theorems at appropriate points in the text. The proofs of these and other of the more difficult theorems are given, but in appendices for the sake of clarity of exposition.

The reviewer feels, however, that this book will be too difficult for the student who has just emerged from the first course in the calculus as given in most of our colleges. Frequent use is made of results ordinarily not met until a second calculus course is taken, although, where this is done, a statement to this effect is made or a reference given to material in the appendices. In several instances results are borrowed from more advanced differential equation theory, and in two cases reference is made to the theory of differential equations in the complex domain. This is presumably done for the purpose of presenting a