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AN APPROXIMATE SOLUTION TO THE NAVIER-STOKES EQUATIONS*

BY MORTON MITCHNER (*Harvard University*)

The purpose of this note is to show how a new approximate solution of the Navier-Stokes equations may be constructed from any given exact solution having a certain specified form. We shall suppose that we are given an exact solution of the Navier-Stokes equations for an incompressible viscous fluid having a velocity field $\mathbf{q}' = (q_1', q_2', q_3') = (u', v', w')$ specified in the form

$$u' = \alpha U(y, z, t),$$
$$v' = \alpha V(y, z, t), \tag{1}$$
$$w' = \alpha W(y, z, t).$$

U, V, and W denote three functions of the position vector $\mathbf{r} = (x_1, x_2, x_3) = (x, y, z)$, and the time coordinate t; α denotes a dimensionless constant. For consistency with the equations of motion (upon taking the divergence of the Navier-Stokes equation, and employing the continuity condition), the pressure p' (and density ρ) must satisfy

$$-\frac{1}{\rho}\nabla^2 p' = \sum_{i,k} \frac{\partial q_i'}{\partial x_k}\frac{\partial q_k'}{\partial x_i}.$$

Hence, it is sufficient to assume that the pressure field has the form

$$p' = \alpha^2 P(y, z, t). \tag{2}$$

Assuming the existence of the above exact solution, we can construct a new approximate solution $[\mathbf{q} = (u, v, w), p]$ of the equations of motion for an incompressible viscous fluid, and this solution is given by

$$u = u_0(a + by) + u' - (t - t_0)bu_0v',$$
$$v = v',$$
$$w = w', \tag{3}$$
$$p = \text{constant},$$

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(u_0, a, b, and t_0 are constants). This solution is valid to first order in α for $t \geqq t_0$ under the hypotheses that

$$\left.\begin{array}{ll} \text{(a)} & \text{the spatial derivatives of } \mathbf{q}' \text{ are bounded for } t \geqq t_0 \\ \text{(b)} & \alpha \ll 1, \\ \text{(c)} & \alpha\, b u_0\, (t - t_0) \ll 1. \end{array}\right\} \quad (4)$$

The velocity field $\mathbf{q}(y, z, t)$ may be regarded as the subsequent time development of an initial disturbance $\mathbf{q}'(y, z, t_0)$ superimposed upon a Couette-type shear flow at time t_0.

The validity of the preceding statement may be checked by direct substitution into the Navier-Stokes equations. Thus, for the u component

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

and hence, employing (3),

$$\frac{\partial u'}{\partial t} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} - bu_0(t - t_0)\left[\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z}\right] = \nu\nabla^2 u' - bu_0(t - t_0)\nu\nabla^2 v'. \quad (5)$$

In virtue of the fact that u' and y' are exact solutions of the Navier-Stokes equations, Eq. (5) states that

$$bu_0(t - t_0)\frac{1}{\rho}\frac{\partial p'}{\partial y} = 0. \quad (6)$$

But $p' = \alpha^2 P(y, z, t)$ and for $t \geqq t_0$, $\partial P/\partial y$ is bounded. The term on the left side of Eq. (6) will therefore be of second order in α provided t also satisfies the condition

$$\alpha^2 bu_0(t - t_0) \ll \alpha, \qquad \text{or} \qquad \alpha bu_0(t - t_0) \ll 1.$$

In a similar fashion, it may also be shown by direct substitution that v and w satisfy the Navier-Stokes equations.

A particular example of the above general result is provided by the known velocity field describing the decay and diffusion of an infinitely long vortex filament initially concentrated on the x axis.

$$\begin{aligned} u' &= 0, \\ v' &= -\frac{K}{2\pi} z \frac{[1 - \exp(-r^2/4\nu t)]}{r^2}, \\ w' &= \frac{K}{2\pi} y \frac{[1 - \exp(-r^2/4\nu t)]}{r^2}, \qquad r^2 = y^2 + z^2. \end{aligned} \quad (7)$$

K denotes the initial circulation or strength of the vortex filament, and is to be associated in its dimensionless form, Kb/u_0, with the parameter of smallness α. For any $t_0 \neq 0$, the solution (7) satisfies the hypotheses (4), and consequently Eqs. (3) and (7) describe the behavior [for $t - t_0 \geqq 0$, but $(t - t_0) \ll 1/bu_0\alpha$] of a vortex filament superimposed on a Couette-type shear flow, the vortex filament being aligned with the direction of flow.

The particular solution stated above exhibits some interesting properties as regards the exchange of vorticity between components. We note that

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} = \xi',$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} = -(t - t_0)bu_0 \frac{\partial v'}{\partial z},$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} = -u_0 b + (t - t_0)bu_0 \frac{\partial v'}{\partial y}.$$

The vorticity associated with the vortex filament disturbance, ξ', remains completely unaffected by the presence of the shear flow and proceeds to decay as if the shear flow were absent. However, the η component of the vorticity, initially zero, begins to grow at the expense of the vorticity of the shear flow. The action of the ξ component is thus analogous to that of a chemical catalyst; while ξ itself remains unaffected by the shear flow, it causes a production of η, drawing upon the infinite field of oriented vorticity in the shear flow. It may be shown that this phenomenon (that ξ is unaffected by the presence of the shear flow) is actually independent of the specific form of the shear flow.

*APPENDIX**

In connection with the preceding remarks, the editors of the Quarterly have brought to the author's attention an investigation by Berker.** Although there is no direct connection between the present work and that of Berker, there does exist a superficial similarity which may be worthy of clarification.

Berker assumes that an exact solution of the Navier-Stokes equations for an incompressible viscous fluid with respect to an inertial frame of reference $oxyz$ is provided by the velocity field $\mathbf{q}'(x, y, z, t)$. Using this given vector point function, Berker then defines a vector field $\mathbf{q}r(X, Y, Z, t) = \mathbf{q}'(X, Y, Z, t)$ with respect to a moving frame of reference $OXYZ$. Corresponding to the motion $\mathbf{q}r(X, Y, Z, t)$ with respect to $OXYZ$, there will exist a motion $\mathbf{q}_B(x, y, z, t)$ with respect to $oxyz$. Berker then determines the conditions under which $\mathbf{q}_B(x, y, z, t)$ will be an exact solution of the Navier-Stokes equations for an incompressible viscous fluid (his Eq. 6.14) and indicates the construction of $\mathbf{q}_B(x, y, z, t)$ in terms of $\mathbf{q}'(x, y, z, t)$ (his Eq. 6.13).

For the particular form of the assumed initial exact solution provided by Eq. (1), Berker's new exact solution has the form

$$\begin{aligned} u_B(y, z, t) &= \dot{a}(t) + u'(Y, Z, t), \\ v_B(y, z, t) &= \dot{b}(t) - (z - c)\Omega + v'(Y, Z, t) \cos \Omega t - w'(Y, Z, t) \sin \Omega t, \\ w_B(y, z, t) &= \dot{c}(t) + (y - b)\Omega + v'(Y, Z, t) \sin \Omega t + w'(Y, Z, t) \cos \Omega t, \end{aligned} \tag{8}$$

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**A. R. Berker, *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible,* Institut de Mécanique des Fluides de L'Université de Lille, 1936.

where Ω is a constant, where $a(t)$, $b(t)$, $c(t)$ are arbitrary functions of t, and where

$$Y = (y - b) \cos \Omega t + (z - c) \sin \Omega t,$$

$$Z = -(y - b) \sin \Omega t + (z - c) \cos \Omega t.$$

Comparison of (3) with (8) makes quite evident that these two solutions are essentially different. Whereas (8) is exact, (3) is approximate. Furthermore, (3) cannot be derived as an approximation from (8).

A MEASURE OF THE AREA OF A HOMOGENEOUS RANDOM SURFACE IN SPACE*

By STANLEY CORRSIN (*Aeronautics Department, The Johns Hopkins University*)

Introduction. We are given an indefinitely large space containing random surface or surfaces homogeneously located in the mean. The problem is to relate the average area of surface per unit volume of space to a simpler statistical quantity, in particular the average number of cuts per unit length made by a straight randomly directed sampling line with the surface.

The plane case will be studied first. After the three dimensional case, illustrative application will be made to the problem of extending to two and three dimensional variables a theorem of S. O. Rice on the average rate of occurrence of any particular value of a one dimensional random variable. Possible use in describing fluid mixing is also indicated.

Two dimensions. Given a plane "homogeneously" inscribed with contour or contours of arbitrary shape. The homogeneity is statistical, i.e. any statistical function associated with the contours is invariant to a translation of coordinate system in the plane. Let $\mathfrak{L}$ be the average contour length enclosed in unit area and let n be the average number of cuts per unit length made by an arbitrary straight traverse line crossing the plane. For a non-isotropic field n is averaged over all traverse directions with uniform weighting; for an isotropic field, any single line will do.

Draw a "very large" square in the plane, L on a side, and subdivide it into "very narrow" traverse strips parallel to one pair of sides.

"Very large" here denotes L so large that averages over L or L^2 are satisfactorily close to their asymptotic values. For example it requires that each traverse strip cross the contours a very large number of times and that the length of contour in L^2 divided by L^2 be as close as we like to $\mathfrak{L}$. "Very narrow" denotes δ so small that virtually all of the intercepted contour segments in a strip can be approximated by secants. This gives restriction on the permissible number of corners and contour intersections.

We imagine the square and strip structure rotated through 180° for averaging purposes in case the field is not isotropic. Then the average number of crossings in one

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