

ON THE DEFORMATION OF ELASTIC SHELLS OF REVOLUTION*

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1. Introduction. In a recent paper, E. Reissner [1] formulated a theory for finite deformation of elastic isotropic shells of revolution where the theory of small deformation (linear theory) is also discussed. In the present note, there is derived a single complex differential equation for small deformation of shells of revolution which is valid for uniform thickness, as well as for a large class of variable thickness.

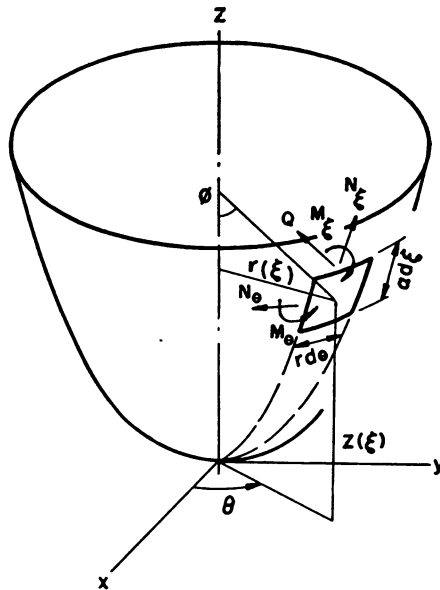


FIG. 1

With the use of cylindrical coordinates r , θ , z , the parametric equation of the middle surface of the shell (see Fig. 1) may be represented by

$$r = r(\xi), \quad z = z(\xi). \quad (1.1)$$

Denoting by ϕ the inclination of the tangent to the meridian of the shell, then

$$r' = \alpha \cos \phi, \quad z' = \alpha \sin \phi, \quad (1.2)$$

where

$$\alpha = [(r')^2 + (z')^2]^{1/2} \quad (1.3)$$

and prime denotes differentiation with respect to ξ .

We note for future reference that the principal radii of curvature r_1 and r_2 are, re-

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spectively, the radius of the curvature of the curve generating the middle surface and the length of the normal intercepted between this curve (generating curve) and the axis of rotation. It follows from the geometry of the middle surface that

$$r = r_2 \sin \phi. \quad (1.4)$$

The stress resultants N_ξ , N_θ and Q , and the stress couples M_ξ and M_θ , acting on an element of the shell, are shown in Fig. 1. Also, as in [1], it is convenient to introduce "horizontal" and "vertical" stress resultants, H and V , given by

$$\alpha N_\xi = r'H + z'V, \quad \alpha Q = -z'H + r'V. \quad (1.5)$$

We now record the basic equations of the small deflection theory of elastic shells of revolution with axisymmetric loading, as given by Reissner in [1].

$$\begin{aligned} rV &= -\int r\alpha p_r d\xi, \\ \alpha N_\theta &= (rH)' + r\alpha p_H, \\ rN_\xi &= (rH) \cos \phi + (rV) \sin \phi, \\ rQ &= -(rH) \sin \phi + (rV) \cos \phi, \\ M_\xi &= \frac{D}{\alpha} \left[\beta' + \nu \frac{r'}{r} \beta \right], \\ M_\theta &= \frac{D}{\alpha} \left[\frac{r'}{r} \beta + \nu \beta' \right], \\ u &= \frac{\tau}{Eh} (N_\theta - \nu N_\xi), \\ w &= \int \left[\frac{z'}{C} (N_\xi - \nu N_\theta) - r'\beta \right] d\xi, \end{aligned} \quad (1.6)$$

where β is the negative change in ϕ due to deformation; u and w are the components of displacement in the radial and axial directions; p_H and p_V denote the components of load intensity in r and z directions; h is the thickness of the shell; and

$$C = Eh, \quad D = \frac{Eh^3}{12(1 - \nu^2)}, \quad (1.7)$$

E and ν being Young's modulus and Poisson's ratio, respectively.

With β and rH as basic variables, proper elimination between Eqs. (1.6), differential equations of equilibrium and compatibility, leads to the following two second-order differential equations:

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)} \beta' - \left[\left(\frac{r'}{r} \right)^2 - \nu \frac{(r'D/\alpha)'}{(rD/\alpha)} \right] \beta + \frac{z'}{(rD/\alpha)} (rH) = \frac{r'}{(rD/\alpha)} (rV), \quad (1.8)$$

$$\begin{aligned} (rH)'' + \frac{(r/\alpha C)'}{(r/\alpha C)} (rH)' - \left[\left(\frac{r'}{r} \right)^2 + \nu \frac{(r'/\alpha C)'}{(r/\alpha C)} \right] (rH) \\ - \frac{z'}{(r/\alpha C)} \beta = \left[\frac{r'z'}{r^2} + \nu \frac{(z'/\alpha C)'}{(r/\alpha C)} \right] (rV) + \nu \frac{z'}{r} (rV)' \\ - \left[\frac{(r/\alpha C)'}{(r/\alpha C)} + \nu \frac{r'}{r} \right] (r\alpha p_H) - (r\alpha p_H)'. \end{aligned} \quad (1.9)$$

2. Normal form of the differential equations. Substitution of the quantities C and D from (1.7) into (1.8) and (1.9) and rearrangement of terms result in

$$L_1(\beta) + \nu \left[\frac{(r'/\alpha)'}{(r/\alpha)} + 3 \frac{(r'h')}{(rh)} \right] \beta + \frac{\alpha^2 m}{r_2 h_0} \left(\frac{h_0}{h} \right) \psi = \frac{\alpha^2 m}{r_2 h_0} \left(\frac{h_0}{h} \right) \left(\frac{mrV}{Eh^2} \right) \cot \phi, \tag{2.1}$$

$$L_1(\psi) - \nu \left[\frac{(r'/\alpha)'}{(r/\alpha)} + 3 \frac{(r'h')}{(rh)} \right] \psi - \frac{\alpha^2 m}{r_2 h_0} \left(\frac{h_0}{h} \right) \beta + 2 \left[\frac{h''}{h} + 2\nu \frac{(r'h')}{(rh)} + \frac{(r/\alpha)'}{(r/\alpha)} \frac{h'}{h} \right] \psi = Z \frac{m}{Eh^2}, \tag{2.2}$$

where Z denotes the right-hand side of (1.9), h_0 is the value of h at some reference section (say $\xi = \xi_0$), and

$$L_1(\) \equiv (\)'' + \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3 \frac{h'}{h} \right] (\)' - \left(\frac{r'}{r} \right)^2 (\), \tag{2.3}$$

$$\psi = \frac{mrH}{Eh^2}, \quad m = [12(1 - \nu^2)]^{1/2}.$$

In (2.1) and (2.2), let

$$\frac{\alpha^2 m}{r_2 h_0} = 2\mu^2 f(\xi), \tag{2.4}$$

where μ is constant and it is to be noted that $f(\xi)$ is independent of the thickness $h(\xi)$. Then, multiplication throughout (2.1) and (2.2) by $\{h[h_0 f(\xi)]^{-1}\}$ results in

$$L(\beta) + \nu\lambda\beta + 2\mu^2\psi = F, \tag{2.5}$$

$$L(\psi) - (\nu\lambda - \delta)\psi - 2\mu^2\beta = G, \tag{2.6}$$

where

$$L(\) \equiv \left[\frac{h_0}{h} f(\xi) \right]^{-1} L_1(\),$$

$$\lambda = \left[\frac{h_0}{h} f(\xi) \right]^{-1} \left\{ \frac{(r'/\alpha)'}{r/\alpha} + \frac{3r'h'}{rh} \right\},$$

$$\delta = 2 \left[\frac{h_0}{h} f(\xi) \right]^{-1} \left\{ \frac{h''}{h} + 2\nu \frac{(r'h')}{(rh)} + \frac{(r/\alpha)'}{(r/\alpha)} \frac{h'}{h} \right\}, \tag{2.7}$$

$$F = 2\mu^2 \frac{mrV}{Eh^2} \cot \phi,$$

$$G = \frac{m}{Eh^2} \left[\frac{h_0}{h} f(\xi) \right]^{-1} Z.$$

Introducing the complex function

$$U = \beta + ik\psi; \quad i = \sqrt{-1} \tag{2.8}$$

where k , an arbitrary function of ξ is to be determined, the differential equations (2.5) and (2.6) may be combined to read

$$L(U) = 2\mu^2 \left(ik - \frac{\nu\lambda}{2\mu^2} \right) \left\{ \beta + \frac{\left[ik \left(\frac{\nu\lambda}{2\mu^2} - \frac{\delta}{2\mu^2} \right) - 1 \right]}{\left(ik - \frac{\nu\lambda}{2\mu^2} \right)} \psi \right\} + i \left[\frac{h_0}{h} f(\xi) \right]^{-1} \left\{ k'' + \left[2 \frac{\psi'}{\psi} + \frac{(r/\alpha)'}{(r/\alpha)} + 3 \frac{h'}{h} \right] k' \right\} \psi + (F + ikG). \tag{2.9}$$

Taking k in the form

$$k = -i \frac{1}{2\mu^2} \left(\nu\lambda - \frac{\delta}{2} \right) + \left\{ 1 - \left[\frac{1}{2\mu^2} \left(\nu\lambda - \frac{\delta}{2} \right) \right]^2 \right\}^{1/2} \tag{2.10}$$

with the restriction (the implication of this restriction will be discussed later) that

$$k' = k'' = 0, \tag{2.11}$$

(2.9) transforms into

$$L(U) = 2\mu^2 \left(ik - \frac{\nu\lambda}{2\mu^2} \right) U + (F + ikG). \tag{2.12}$$

By putting the last complex differential equation in the form

$$L_1(U) - i \frac{2\mu^2 h_0 \left(k + i \frac{\nu\lambda}{2\mu^2} \right)}{h f^{-1}(\xi)} U = \frac{h_0}{h f^{-1}(\xi)} (F + ikG)$$

and observing that the coefficient of U' , resulting from the application of the operator L_1 , defined by (2.3), is

$$R = \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3 \frac{h'}{h} \right],$$

and that

$$\exp \left[\frac{1}{2} \int R d\xi \right] = h^{3/2} \left(\frac{r}{\alpha} \right)^{1/2}$$

then, with the aid of the transformation

$$W = \left(\frac{h}{h_0} \right)^{3/2} \left(\frac{r}{\alpha} \right)^{1/2} U, \tag{2.13}$$

we obtain

$$W''' + [2i^3 \mu^2 \Psi^2(\xi) + \Lambda(\xi)] W = \left[\frac{h}{h_0} \frac{r}{\alpha} \right]^{1/2} f(\xi) [F + ikG] \tag{2.14}$$

which is the normal form of (2.12), and where

$$\Psi^2 = \left(k + i \frac{\nu\lambda}{2\mu^2} \right) \left(\frac{h_0}{h} \right) f(\xi),$$

$$\Lambda = -\frac{1}{2} \frac{(r/\alpha)''}{(r/\alpha)} + \frac{1}{4} \left[\frac{(r/\alpha)'}{(r/\alpha)} \right]^2 - \left(\frac{r'}{r} \right)^2 - \frac{3}{2} \frac{(r/\alpha)'}{(r/\alpha)} \frac{h'}{h} - \frac{3}{2} \frac{h''}{h} - \frac{3}{4} \left(\frac{h'}{h} \right)^2. \tag{2.15}$$

We now return to (2.12) and observe that condition (2.11) is fulfilled only if k is a constant, and this may be achieved by proper choice of λ and δ . In particular, we note the following two cases.

(a) For shells of variable thickness and with reference to differential equation (2.12), the condition (2.11) is satisfied, provided $(\nu\lambda - \delta/2)$ is constant. Thus, by (2.7),

$$\left(\frac{r}{\alpha}\right)h'' + \left(\frac{r}{\alpha}\right)'h' - \nu\left[\left(\frac{r'}{\alpha}\right)h' + \left(\frac{r'}{\alpha}\right)'h\right] = \left(\frac{r}{\alpha}\right)Kf(\xi). \quad (2.16)$$

This equation is directly integrable and its solution is

$$h = Kr^{\nu} \int r^{-(1+\nu)} \alpha \left[\int \left(\frac{r}{\alpha}\right) f d\xi \right] d\xi + c_1 r^{\nu} + c_2 r^{\nu} \int r^{-(1+\nu)} \alpha d\xi, \quad (2.17)$$

where c_1, c_2 are constants of integration. It may be noted that setting $K = 0$ corresponds to the vanishing value of $(\nu\lambda - \delta/2)$ or, by (2.10), to $k = 1$.

(b) For shells of uniform thickness, δ vanishes identically and we have

$$\lambda = f^{-1}(\xi) \left[\frac{(r'/\alpha)'}{(r/\alpha)} \right], \quad \delta = 0. \quad (2.18)$$

Clearly, λ is a function of ξ and its form is determined by the geometry of the middle surface. However, for numerous shell configurations λ and, by (2.10), k is either exactly or very nearly a constant.

It should be mentioned that whenever the radius of curvature of the generating curve r_1 is a constant (r_2 may be a function of ξ), then, with proper choice of ξ ($\xi = \phi$) and by (1.2), (2.4) and (2.18), λ , and thus k , are in fact constant. The cases of conical shell and toroidal shell, treated recently by Clark [2], are included in this class.

3. Remarks on the solution of equation (2.14). Particular solutions of Eq. (2.14) may be obtained approximately by the membrane theory of shells given by Hildebrand in [3] or sometimes by a more recent method developed by Clark and Reissner [4]. In this section, we shall discuss briefly the homogeneous solution of equation (2.14).

If Ψ^2 and Λ are suitably regular over a finite interval of the ξ -axis and furthermore, if Ψ^2 is bounded from zero everywhere within this interval, then the classical method of asymptotic integration leads at once to the solution

$$W = \Psi^{-1/2} \{ A e^{-i\eta} + B e^{i\eta} \} \quad (3.1)$$

which, by means of well-known relations for Bessel functions, may also be written as

$$W = \left[\Psi^{-1} \int \Psi d\xi \right]^{1/2} \{ A J_{-1/2}(\eta) + B J_{1/2}(\eta) \}, \quad (3.2)$$

where A and B are constants and

$$\eta = (2i^3)^{1/2} \mu \int \Psi d\xi. \quad (3.3)$$

According to Langer [5], if Ψ^2 is not bounded from zero everywhere within the interval in question but vanishes to the degree n at some point ξ_0 within this interval, then (3.2)

may be generalized to

$$W = \left[\Psi^{-1} \int_{\xi_0}^{\xi} \Psi(\zeta) d\zeta \right]^{1/2} \{ A J_{-1/n+2}(\eta) + B J_{1/n+2}(\eta) \} \quad (3.4)$$

which is valid at ξ_0 .

In (3.4)
$$\frac{\xi}{\eta} = (2i^3)^{1/2} \mu \int_{\xi_0}^{\xi} \Psi(\zeta) d\zeta.$$

An appropriate form of this solution was recently employed for toroidal shells by Clark [2].

If, moreover, the coefficient function Λ of the differential equation has a pole of second order at ξ_0 , then by a more recent method of asymptotic integration developed by Langer [6], a representation of W in terms of Bessel functions is again possible and is valid at ξ_0 . It may be mentioned that application of this method yields a solution for ellipsoidal shells of revolution which is valid at the apex (where a pole of second order occurs); this will be given on another occasion.

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REFERENCES

1. Eric Reissner, *On the theory of thin elastic shells*, H. Reissner Anniv. Vol., 1949, pp. 231-247
2. R. A. Clark, *On the theory of thin elastic toroidal shells*, J. Math. Phys. 29, 146-178 (1950)
3. F. B. Hildebrand, *On asymptotic integration in shell theory*, Proc. 3rd. Symp. in Appl. Math. 3, 53-66 (1950)
4. R. A. Clark and E. Reissner, *Bending of curved tubes*, Advances in Applied Mechanics II, Academic Press, 1950, pp. 93-122
5. R. E. Langer, *On the asymptotic solution of ordinary differential equations*, Trans. Am. Math. Soc. 33, 23-64 (1931)
6. R. E. Langer, *On the asymptotic solution of ordinary differential equations, with reference to the Stokes' phenomenon about a singular point*, Trans. Am. Math. Soc. 37, 397-416 (1935)