QUARTERLY OF APPLIED MATHEMATICS

Vol. XIII JULY, 1955 No. 2

THE ASYMPTOTIC BOUNDARY LAYER ON A CIRCULAR CYLINDER IN AXIAL INCOMPRESSIBLE FLOW*

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1. Introduction. If a fluid is streaming past a body of revolution with a velocity at infinity parallel to the generators of the body, then it is well-known [2] that the boundary layer induced on it is equivalent to the boundary layer on a flat plate provided only that the thickness of the layer is small compared with the radius of the body. Recently an increasing amount of attention has been paid to the flow of compressible fluids over slender bodies of revolution at high speeds. Since the displacement thickness δ_1 of the boundary layer on a flat plate is of the order $M^2xR^{-1/2}$ where x is the distance from the leading edge, M is the Mach number of the flow in the main stream and R is the Reynolds number based on x, there is a real possibility that δ_1 may be at least of the same order as the radius of the body. It is of interest to examine the modification to the boundary layer which is then necessary. We restrict attention to incompressible flow, but it is hoped that the methods developed here may be extended to compressible boundary layers.

The boundary layer on the outside of a circular cylinder near to the leading edge has already been considered by R. A. Seban and R. Bond [3]. They assumed that the axis of cylindrical tube was parallel to the direction of flow and that the boundary layer was of zero thickness at the leading edge of the cylinder. They expanded the stream function in a power series in ascending powers of $(x/a)^{1/2}$ where x is the axial distance from the leading edge and a the radius of the cylinder. The leading term is the Blasius solution for a flat plate in a uniform stream and they computed numerically the two following terms. Among other things they found that the skin friction coefficient C_f on the cylinder is

$$0.664\left(\frac{\nu}{xU}\right)^{1/2}\left[1 + 0.53\left(\frac{16x\nu}{Ua^2}\right)^{1/2} - 0.071\left(\frac{16x\nu}{Ua^2}\right) + \cdots\right],$$

where ν is the coefficient of kinematic viscosity and U the (constant) velocity in the main stream. The initial effect of curvature is therefore to increase the skin friction.

It is the aim of this paper to examine the boundary layer at large values of x. For this purpose we are not concerned with the precise form of the body near $x = 0$, although in fact we shall assume it to be a circular cylinder for all $x \geq 0$, and little modification

*Received March 8, 1954.

would be needed to account for the effect of a rounded nose. We show that if the velocity of the main stream is constant, the skin friction may be expanded as a double power series in descending powers of $\log \xi$ and ξ of which the leading term is proportional to $[\log \xi]^{-1}$ where $\xi = 2x\nu/Ua^2$. It is thus of larger order than at corresponding points of a flat plate, so that the increase found by Seban and Bond [3] near the leading edge is maintained a long way downstream. The thickness of the boundary layer is, however, only slightly reduced in comparison with the flat plate and this implies that most of the changes in velocity occur relatively close to the cylinder.

The asymptotic form of the boundary layer is also considered when the main stream velocity is proportional to x^m where $m < 1$. If $m > 1$ the boundary layer thickness diminishes as x increases so that the effect of curvature is small when x/a is large, and if $m = 1$ the boundary layer is of constant thickness and there is a solution dependent on one parameter only [1]. We show that if $-\frac{1}{2} < m < 1$ the leading term in the asymptotic expansion of the skin friction is also proportional to $[\log \xi]^{-1}$ and we infer that the effect of the curvature of the body is to delay separation.

The contrast between the boundary layers on a flat plate and on a circular cylinder is not confined to the skin friction, for on the flat plate the boundary layer, while diffusing outwards, retains a similar form, whereas on the circular cylinder the form is always altering and ultimately tends to that given by Oseen's approximation. The difference between plane and axially symmetric boundary layers may also be illustrated as follows. Suppose we have a viscous fluid occupying the region between two parallel planes a distance d apart, of which one is at rest and the other moves with velocity U parallel to itself. Then the velocity u of the fluid is given by

$$u = U(1 - y/d),$$

where y measures distance from the moving plane. It is seen that the form of the velocity profile is independent of d, being always a straight line. Now consider the corresponding problem in axial flow. We have a pipe bounded by two concentric cylinders of which the outer of radius b moves parallel to itself with uniform velocity U and the inner of radius a is at rest. Then the velocity u of the fluid in the pipe is

$$u = U \frac{\log (r/a)}{\log (b/a)},$$

where r measures distance from the axis. It is seen that the radii of the cylinders are of great importance in determining the form of the velocity profile. For example if we fix b and r, then as $a \to 0$, $u \to U$ so that for small a there is a boundary layer in the vicinity of the inner cylinder. If we interpret these two solutions as the asymptotic form of the boundary layers when the fixed surface is semi-infinite, it may be deduced that in the two-dimensional problem the boundary layer on the fixed surface grows until it has spread uniformly throughout the fluid, while in the axially symmetric case with a small, it is always confined to the neighbourhood of $r = a$.

2. The equations of the boundary layer. We consider a circular cylinder of radius a whose axis occupies the positive half of the x-axis. Let r denote distance from this axis and u, v the components of the fluid velocity along the directions of x and r respectively. Then, if ν is the coefficient of kinematic viscosity, p the pressure, and ρ the density, the equations of motion in the incompressible boundary layer are

$$\left.\begin{aligned} \frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \\ \text{and} \qquad o(1) &= -\frac{1}{\rho}\frac{\partial p}{\partial r}. \end{aligned}\right\} \tag{2.1}$$

The boundary conditions are

$$u = v = 0 \qquad \text{when } r = a,\ x > 0,$$

$$u \to U(x) \text{ as } r \to \infty \text{ for fixed } x \text{ and as } x \to 0+ \text{ for fixed } r > a,$$

where $U(x)$ is the velocity of the fluid in the main stream. The third equation may be interpreted as implying that the pressure variation is small across the boundary layer. We shall assume this to begin with, and then, using the solution we obtain, it may be shown that the pressure variation across the boundary layer is $O(\rho U\nu/x)$ and negligible when x is large. The values of U in which we shall be interested in this paper will be proportional to x^m so that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{mU^2}{x} \tag{2.2}$$

and from the equation of continuity we can define a stream function

$$\nu x\Psi(\xi, \eta),$$

where

$$\xi = 2x\nu/Ua^2 \qquad \text{and} \qquad \eta = r^2U/2x\nu. \tag{2.3}$$

Then

$$u = \frac{1}{r}\frac{\partial}{\partial r}(\nu x\Psi) = U\frac{\partial \Psi}{\partial \eta} \tag{2.4}$$

and

$$v = -\frac{1}{r}\frac{\partial}{\partial x}(\nu x\Psi) = -\frac{\nu}{r}\left[\Psi - (1-m)\eta\frac{\partial\Psi}{\partial\eta} + (1-m)\xi\frac{\partial\Psi}{\partial\xi}\right]. \tag{2.5}$$

The equation satisfied by Ψ is

$$2\eta\frac{\partial^3\Psi}{\partial\eta^3} + (2+\Psi)\frac{\partial^2\Psi}{\partial\eta^2} + m\left[1 - \left(\frac{\partial\Psi}{\partial\eta}\right)^2\right] = (1-m)\xi\left(\frac{\partial\Psi}{\partial\eta}\frac{\partial^2\Psi}{\partial\eta\,\partial\xi} - \frac{\partial\Psi}{\partial\xi}\frac{\partial^2\Psi}{\partial\eta^2}\right). \tag{2.6}$$

The boundary conditions are

$$\frac{\partial\Psi}{\partial\eta} = \Psi + (1-m)\xi\frac{\partial\Psi}{\partial\xi} = 0 \qquad \text{at} \qquad \eta\xi = 1, \tag{2.7}$$

$$\partial\Psi/\partial\eta \to 1 \qquad \text{as} \qquad \eta \to \infty \qquad \text{for all } \xi \qquad \text{and as} \qquad \xi \to 0+,$$

provided $\eta\xi > 1$.

We note that if $m = 1$ then η is a function of r only and that a solution may be found with Ψ a function of η only. This special case has already been discussed by Cooper and Tulin [1]. If $m > 1$, then $\xi \to 0$ as $x \to \infty$. The boundary layer decreases in thickness as $x \to \infty$ so that the effects of curvature are greatest at the leading edge and may be neglected in the asymptotic expansion, which may therefore be reduced to that for a flat plate with the same mainstream velocity.

3. Uniform mainstream. Here $m = 0$ and to obtain the first and crucial term of the asymptotic series for Ψ we assume that as $\xi \to \infty$, $\xi \partial\Psi/\partial\xi \to 0$ for fixed η. The right hand side of (2.6) is then zero in the limit $\xi \to \infty$ and the boundary conditions reduce to

$$\Psi = \frac{\partial\Psi}{\partial\eta} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \frac{\partial\Psi}{\partial\eta} \to 1 \quad \text{as} \quad \eta \to \infty.$$

Equation 2.6 may be integrated once to give

$$\eta \frac{\partial^2\Psi}{\partial\eta^2} = A \exp\left[-\int_0^\eta \frac{\Psi\, d\eta}{2\eta}\right], \tag{3.1}$$

where A is a constant. We deduce that either $\partial^2\Psi/\partial\eta^2 \sim 1/\eta$ near $\eta = 0$ which implies that the boundary condition on $\partial\Psi/\partial\eta$ at $\eta = 0$ is violated or $A = 0$ and $\partial^2\Psi/\partial\eta^2$ except possibly at $\eta = 0$. We accept the second possibility and obtain as our first approximation

$$\Psi = \eta. \tag{3.2}$$

This is simply the stream function of the undisturbed stream and satisfies (2.6). The boundary condition at $\eta = \xi^{-1}$ is invalidated however. The approximation may be improved by substituting back into (3.1) and applying the boundary condition on $\partial\Psi/\partial\eta$ at $\eta = \xi^{-1}$ instead of at $\eta = 0$. We find that

$$\eta \frac{\partial^2\Psi}{\partial\eta^2} = Ae^{-\eta/2},$$

whence

$$\frac{\partial\Psi}{\partial\eta} = 1 - A\int_\eta^\infty \frac{e^{-z/2}\, dz}{z}.$$

Now when η is small

$$\int_\eta^\infty \frac{e^{-z/2}\, dz}{z} = -\log\frac{\eta C}{2} + \sum_0^\infty \frac{(-)^n}{(n+1)(n+1)!}\left(\frac{\eta}{2}\right)^{n+1}, \tag{3.3}$$

where $\log C = 0.577 \cdots$ and is Euler's constant.
Hence, since $\partial\Psi/\partial\eta = 0$ at $\eta\xi = 1$,

$$A = [\log(2\xi/C)]^{-1} + O[\xi \log(2\xi/C)]^{-1}, \tag{3.4}$$

the second approximation to $\partial\Psi/\partial\eta$ is

$$1 - \frac{1}{\log(2\xi/C)}\int_\eta^\infty \frac{e^{-z/2}\, dz}{z}, \tag{3.5}$$

and we note that the correction term just found is small when ξ is large except near $\eta\xi = 1$. This suggests that we write

$$\Psi = \eta + \sum_{s=1}^\infty \frac{F_s(\eta)}{[\log(2\xi/C)]^s} + O\left(\frac{1}{\xi\log\xi}\right) \tag{3.6}$$

and investigate the properties of $F_s(\eta)$. If our expansion is to be meaningful then it must be possible to make the error in Ψ as small as we please by taking ξ large enough and taking a sufficient number of terms. Certainly this is not true if we retain the first term only, but we shall show below that the successive terms are of uniformly decreasing order which suggests that our requirement is met.

It is of interest to note that the first two terms of this expansion are the same as those obtained on Oseen's approximation when x is large and it is likely that a similar result would be found whatever the cross section of the cylinder. In contradistinction to the boundary layer on the flat plate, it appears in fact that Oseen's approximation becomes more accurate as x increases.

Substituting in the differential equation and comparing coefficients of $(\log 2\xi/C)^{-s}$ we obtain

$$2\eta F_s''' + (2 + \eta)F_s'' = -(s - 1)F_{s-1}' - F_{s-1}F_1'' - \sum_{t=1}^{s-2} [tF_t'F_{s-t-1}' + F_t(tF_{s-t-1}'' - F_{s-t}'')], \tag{3.7}$$

where the primes denote differentiation with respect to η. In the determination of the F_s we shall take as boundary conditions

$$\Psi = 0 \text{ at } \eta = 0, \quad \partial\Psi/\partial\eta = 0 \text{ at } \eta = \xi^{-1} \text{ and } \partial\Psi/\partial\eta \to 1 \text{ as } \eta \to \infty. \tag{3.8}$$

The first of these conditions is inaccurate and so later on we must investigate how large is the error thereby incurred. We now prove that if

$$F_t'(\eta) = D_t \log \tfrac{1}{2}\eta C + E_t + A_t\eta(\log \tfrac{1}{2}\eta C)^2 + B_t\eta \log \tfrac{1}{2}\eta C + O(\eta) \tag{3.9}$$

near $\eta = 0$, for $1 \le t \le s - 1$, where the A's, B's, D's and E's are constants, then $F_s'(\eta)$ has a similar form near $\eta = 0$. It may then be inferred that (3.9) is the correct form for $F_s'(\eta)$ for all $s \ge 1$ since it is the correct form when $s = 1$. From (3.9) it follows that near $\eta = 0$

$$F_t = D_t\eta \log \tfrac{1}{2}\eta C + (E_t - D_t)\eta + \tfrac{1}{2}A_t\eta^2(\log \tfrac{1}{2}\eta C)^2 + \tfrac{1}{2}(B_t - A_t)\eta^2 \log \tfrac{1}{2}\eta C + O(\eta^2),$$

since $F_t = 0$ at $\eta = 0$, and

$$F_t'' = D_t\eta^{-1} + A_t(\log \tfrac{1}{2}\eta C)^2 + (2A_t + B_t) \log \tfrac{1}{2}\eta C + O(1).$$

Hence substituting into (3.7) we find that

$$2\eta F_s''' + (2 + \eta)F_s'' = 2A_s(\log \tfrac{1}{2}\eta C)^2 + 2(B_s + 4A_s) \log \tfrac{1}{2}\eta C + O(1), \tag{3.10}$$

where

$$2A_s = -\sum_{t=1}^{s-2} tD_tD_{s-t-1}$$

and

$$-2B_s - 8A_s = (s - 1)D_{s-1} + D_1D_{s-1} + \sum_{t=1}^{s-2} (tD_tD_{s-t-1} - D_tD_{s-t}) + \sum_{t=1}^{s-2} t(D_tE_{s-t-1} + D_{s-t-1}E_t).$$

Integrating (3.10) once, we obtain

$$\eta F_s'' = A_s\eta(\log \tfrac{1}{2}\eta C)^2 + (B_s + 2A_s)\eta \log \tfrac{1}{2}\eta C + D_s + O(\eta),$$

whence

$$F_s' = D_s \log \tfrac{1}{2}\eta C + E_s + A_s\eta(\log \tfrac{1}{2}\eta C)^2 + B_s\eta \log \tfrac{1}{2}\eta C + O(\eta),$$

and, comparing with (3.9), we see that the desired result is proved. Thus, near $\eta = 0$

$$\frac{\partial \Psi}{\partial \eta} = 1 + \sum_{s=1}^{\infty} \frac{D_s \log \frac{1}{2}\eta C + E_s}{[\log (2\xi/C)]^s} + O[\xi^{-1}[\log (2\xi/C)]^{-1}].$$

In particular, when $\eta = 1/\xi$, $\partial\Psi/\partial\eta = 0$, so that

$$1 = \sum_{s=1}^{\infty} \frac{D_s}{[\log (2\xi/C)]^{s-1}} - \sum_{s=1}^{\infty} \frac{E_s}{[\log (2\xi/C)]^s} \tag{3.11}$$

and therefore

$$D_1 = 1; \qquad D_s = E_{s-1}, \qquad s > 1. \tag{3.12}$$

We may now express A_s, B_s in terms of D_t obtaining

$$2A_s = -\sum_{t=1}^{s-2} tD_tD_{s-t-1} \tag{3.13}$$

and

$$B_s + 4A_s = -(s-1)D_{s-1} - \frac{1}{2}\sum_{t=1}^{s-2} \{2(t-1)D_tD_{s-t} + tD_tD_{s-t-1}\}. \tag{3.14}$$

Since E_s is determined from the condition that $F_s' \to 0$ as $\eta \to \infty$, and A_s, B_s, D_s are dependent only on the E's we may determine as many of them as we please by successive substitution. The procedure is illustrated below by the determination of D_2 and D_3.

The equation of F_1 is

$$2\eta F_1''' + (2 + \eta)F_1'' = 0,$$

with solution

$$F_1'' = D_1\eta^{-1}e^{-\eta/2}, \qquad F_1' = -D_1 \int_\eta^\infty e^{-z/2}\frac{dz}{z}$$

and

$$F_1 = -D_1\eta \int_\eta^\infty e^{-z/2}\frac{dz}{z} - 2D_1(1 - e^{-\eta/2}). \tag{3.15}$$

From (3.3) and (3.13) it now follows that

$$D_1 = 1 \qquad \text{and} \qquad E_1 = 0. \tag{3.16}$$

The equation for F_2 is

$$2\eta F_2''' + (2 + \eta)F_2'' = -F_1' - F_1F_1'',$$

whence

$$\frac{d}{d\eta}(2\eta e^{\eta/2}F_2'') = -\frac{1}{\eta}F_1 - F_1'e^{\eta/2},$$

so that

$$2\eta e^{\eta/2}F_2'' = -\int^\eta F_1'e^{\eta/2}\,d\eta - \int^\eta F_1\frac{d\eta}{\eta}$$
$$= (2e^{\eta/2} + \eta + 2)\int_\eta^\infty e^{-z/2}\frac{dz}{z} + 4\log\eta - 2e^{-\eta/2} + C_2\,, \tag{3.17}$$

where C_2 is a constant.

When η is small the right hand side is equal to

$$-4\log\tfrac{1}{2}\eta C + 4\log\eta - 2 + C_2 = 2D_2\,. \tag{3.18}$$

Hence from (3.12) and (3.16)

$$C_2 = 2 + 4\log\tfrac{1}{2}C$$

and, since $F_2' \to 0$ as $\eta \to \infty$,

$$E_2 = -\frac{1}{2}\int_0^\infty e^{-\eta/2}\frac{d\eta}{\eta}\left\{(2e^{\eta/2} + \eta + 2)\int_\eta^\infty e^{-z/2}\frac{dz}{z} + 4\log\eta - 2e^{-\eta/2} + C_2\right\}$$
$$= -2\log 2 - \frac{\pi^2}{4} = D_3\,. \tag{3.19}$$

Further terms of the series may be found in a similar way although the computations soon become very complicated and recourse to numerical methods is then necessary.

In order to complete this discussion of the asymptotic boundary layer on the circular cylinder we must now investigate the leading error term in (3.6) to show that it is sufficiently small for large x. We write

$$\Psi = \eta + \sum_{s=1}^\infty \frac{F_s(\eta)}{[\log(2\xi/C)]^s} + \Phi \tag{3.20}$$

and suppose that the F_s are all known. The differential equation (2.6) is satisfied if $\Phi \equiv 0$ but not the boundary conditions and, neglecting terms involving η^2 and powers of $\log\frac{1}{2}C\eta$, it follows from (3.9) that near $\eta = 0$

$$F_t' = D_t\log\tfrac{1}{2}\eta C + E_t + A_t\eta(\log\tfrac{1}{2}\eta C)^2 + B_t\eta\log\tfrac{1}{2}\eta C + C_t\eta + \cdots$$

where the A's, B's and C's may be expressed in terms of the D's and E's. Hence when $\eta = \xi^{-1}$ we have, neglecting ξ^{-2},

$$\frac{\partial\Phi}{\partial\eta} = -\sum_{s=1}^\infty\frac{A_s}{\xi[\log(2\xi/C)]^{s-2}} + \sum_{s=1}^\infty\frac{B_s}{\xi[\log(2\xi/C)]^{s-1}} - \sum_{s=1}^\infty\frac{C_s}{\xi[\log(2\xi/C)]^s}$$

and

$$\Phi + \xi\frac{\partial\Phi}{\partial\xi} - \eta\frac{\partial\Phi}{\partial\eta} = \sum_{s=1}^\infty\frac{(s+1)D_s}{\xi[\log(2\xi/C)]^s} + \sum_{s=1}^\infty\frac{s(E_s - D_s)}{\xi[\log(2\xi/C)]^{s+1}}\,. \tag{3.21}$$

Now from (3.13) and (3.14) and the solution for F_s already worked out,

$$A_1 = 0, \quad A_2 = 0, \quad A_3 = -\tfrac{1}{2}, \quad A_4 = 0 \quad \cdots$$
$$B_1 = 0, \quad B_2 = -1, \quad B_3 = 0, \quad \cdots$$
$$C_1 = \tfrac{1}{2}, \quad C_2 = \tfrac{5}{2}, \quad \cdots$$

and hence the boundary conditions for Φ are

$$\frac{\partial \Phi}{\partial \eta} = \frac{1}{\xi[\log(2\xi/C)]} - \frac{5}{2\xi[\log(2\xi/C)]^2} + \cdots,$$

$$\Phi + \xi \frac{\partial \Phi}{\partial \xi} - \eta \frac{\partial \Phi}{\partial \eta} = \frac{2}{\xi[\log(2\xi/C)]} - \frac{2}{\xi[\log(2\xi/C)]^2} + \cdots, \tag{3.22}$$

when $\eta = \xi^{-1}$ and ξ is large, and $\partial\Phi/\partial\eta \to 0$ as $\eta \to \infty$.
If now we put

$$\Phi = \frac{G_1(\eta)}{\xi[\log(2\xi/C)]^2} + O\left(\frac{1}{\xi[\log(2\xi/C)]^3}\right),$$

then $2\eta G_1''' + 2G_1'' + \eta G_1'' = 0$
with the solution

$$G_1'(\eta) = \int_\eta^\infty e^{-z/2} \frac{dz}{z}. \tag{3.23}$$

Further terms of the series for Φ may be obtained if desired but sufficient has been done to show the form which the asymptotic expansion of $\partial\Psi/\partial\eta$ must take, viz.,

$$\frac{\partial \Psi}{\partial \eta} = 1 + \sum_{s,t=1}^{\infty} \frac{P'_{s,t}(\eta)}{\xi^{s-1}[\log(2\xi/C)]^t}, \tag{3.24}$$

where the P's are to be determined successively first for $s = 1$ and all t, then $s = 2$ and all t, etc. It is noted that there are no terms in the expansion of $\partial\Psi/\partial\eta$ with $t = 0$ owing to the form of the expansion in (3.3). Further it is only necessary to obtain the $P_{1,t}(\eta)$ for $1 \leq t \leq p$ in order to determine $P_{2,p-2}$ although such terms are of higher order than any of the $P_{1,t}$. Finally it is pointed out that as soon as we begin to examine the $P_{2,t}$ the effect of the shape of the body near $x = 0$ becomes of importance so that for example a finite shift of the leading cross section of the cylinder along the x axis will change them.

The skin friction on the cylinder

$$\begin{aligned} \mu\left(\frac{\partial u}{\partial r}\right)_{r=a} &= \rho U^2 \frac{a}{x} \left(\frac{\partial^2 \Psi}{\partial \eta^2}\right)_{\eta=1/\xi} \\ &= \rho U^2 \frac{a}{x}\left[\frac{F_1''(\eta)}{\log(2\xi/C)} + \frac{F_2''(\eta)}{[\log(2\xi/C)]^2} + \cdots + \frac{G_1''(\eta)}{\xi[\log(2\xi/C)]^2} + \cdots\right]. \end{aligned} \tag{3.25}$$

Now when η is small

$$F_1''(\eta) = \eta^{-1} - \tfrac{1}{2} + O(\eta); \qquad F_2''(\eta) = -\log \eta C/2 + \tfrac{3}{2} + O(\eta);$$

$$F_3''(\eta) = -\eta^{-1}(2\log 2 + \tfrac{1}{4}\pi^2) - \tfrac{1}{2}(\log \tfrac{1}{2}\eta C)^2 - \log \tfrac{1}{2}\eta C + O(1);$$

$$F_4''(\eta) = \eta^{-1}D_4 + O(\log \eta), \qquad \text{and} \qquad G_1''(\eta) = \eta^{-1} + O(1).$$

Hence after some reduction

$$\begin{aligned} \left(\mu \frac{\partial u}{\partial r}\right)_{r=a} = \frac{2\mu U}{a}\bigg[&\frac{1}{\log(4x\nu/Ca^2U)} - \frac{2\log 2 + \frac{1}{4}\pi^2}{[\log(4x\nu/Ca^2U)]^3} + \cdots \\ &+ \frac{a^2U}{2x\nu}\left\{\frac{7}{2\log(4x\nu/Ca^2U)} + \cdots\right\} + \cdots\bigg]. \end{aligned} \tag{3.26}$$

For a flat plate at a distance x from the leading edge the skin friction is

$$[\mu(\partial u/\partial y)]_{y=0} = 0.332\mu U(U/\nu x)^{1/2},$$

which is of smaller order than (3.26) when $x\nu/a^2U$ is large. Thus the trend noticed by Seban and Bond [3] near the leading edge of the cylinder is emphasized at large distances downstream and we may conclude that the effect of the curvature of the body is to increase the skin friction, especially at large distances downstream when the boundary layer is thicker than the radius of the cylinder. The skin friction is then almost constant, diminishing like $(\log x)^{-1}$.

We may also calculate the velocity profile at large distances from the cylinder when x is large and we find that

$$\frac{u}{U} \approx 1 - \frac{2}{\eta}\frac{e^{-\eta/2}}{\log 2\xi/C}.$$

The boundary layer thickness is ultimately of order

$$x\left(\frac{\nu}{Ux \log (4x\nu/Ca^2U)}\right)^{1/2},$$

so that the effect of curvature on it is small thinning it out by a factor $(\log x)^{-1/2}$.

4. Main stream velocity proportional to x^m. There is a difficulty with the boundary layers caused by main stream velocities of this kind at the leading edge of the cylinder, especially if $m < 0$. We are however interested in the behaviour of the boundary layers at large distances downstream, so that we may avoid the difficulty by supposing that the main stream approaches this form at large distances downstream and our solution will give the asymptotic form for the ensuing boundary layer.

As pointed out in Sec. 2 our method is only applicable if $m < 1$ and then we write

$$\Psi = \eta + \sum_{s,t=1}^{\infty} \frac{P_{s,t}(\eta)}{(\log \xi)^t \xi^{s-1}}, \tag{4.1}$$

with the boundary conditions

$$\frac{\partial \Psi}{\partial \eta} \to 1 \quad \text{as} \quad \eta \to \infty, \qquad \frac{\partial \Psi}{\partial \eta} = \Psi + (1 - m)\xi \frac{\partial \Psi}{\partial \xi} = 0,$$

when $\eta = \xi^{-1}$.

The values of $P_{s,t}$ may be obtained seriatim.

Of particular interest is the effect of curvature on the tendency of the boundary layer to separate, and some information on this score may be obtained from the leading terms of (4.1). If we set $P_{1,1}(\eta) = H(\eta)$ then

$$2\eta H''' + (2 + \eta)H'' - 2mH' = 0,$$

with boundary conditions

$$H' = -\log \xi \quad \text{at} \quad n = \xi^{-1} \quad \text{and} \quad H' \to 0 \quad \text{as} \quad \eta \to \infty.$$

The appropriate solution is

$$H'(\eta) = A \int_1^{\infty} e^{-\eta t/2}\left(\frac{t-1}{t}\right)^{2m} \frac{dt}{t} \tag{4.2}$$

if $2m > -1$. Defined by this integral, $H'(\eta) \to 0$ exponentially as $\eta \to \infty$ and near $\eta = 0$

$$H'(\eta) = -A \log \tfrac{1}{2} \eta C + A \int_0^1 \frac{dt}{t} [(1 - t)^{2m} - 1] + \cdots .$$

Hence $A = -1$ and

$$\mu\left(\frac{\partial u}{\partial r}\right)_{r=a} = \frac{2U\mu}{a \log \xi} + \text{higher order terms.}$$

If $2m = -1$ the integral is singular and in fact the method breaks down because $H' = Ae^{-\eta/2}$ and does not behave logarithmically near $\eta = 0$. It is not clear at the moment what is the correct procedure in this case. Provided $-2m$ is not a positive integer, the method outlined in the previous section is formally adequate but in view of the breakdown at $m = -\frac{1}{2}$ we cannot assume at present that it is physically significant.

Now if the main stream along a flat plate is proportional to x^m we know that there are similar solutions of the boundary layer equations with positive skin friction if $m > -0.0904$ while from our work above we see that on a circular cylinder the skin friction is certainly ultimately positive if $m > -0.5$. We must bear in mind however that the conventional boundary layer comes to an end at separation, and that the same may well be true of a circular cylinder. There is a real danger that separation will occur at a finite value of x if $-0.5 < m < -0.0904$. Nevertheless we may conclude that the effect of the curvature of the cylinder, when the boundary layer has a thickness comparable with its radius of curvature, is to delay separation.

Acknowledgment. The author would like to thank Dr. Lagerstrom, Mr. Lees and Dr. Kaplun for stimulating discussions on the work described in this paper which was done while the author was at the California Institute of Technology on leave of absence from Bristol University. It was supported by the Office of Naval Research (Contract N6 ONR-244, T.O.VIII).

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