Hence, if (11) is satisfied, then

$$\liminf_{t\to\infty} N^2(t)/t^2 = 0.$$

Since this contradicts the necessary condition (5), the proof is complete.

5. In view of the first of the conditions (8), it is worth mentioning what happens to each of the three stability properties, considered above, if (1) is replaced by a differential equation

$$x'' + g(t)x = 0 \tag{13}$$

which is a *small* perturbation of (1), in the following sense:

$$\int^{\infty} | f(t) - g(t) | \, dt < \infty. \tag{14}$$

Under the assumption (14), both (1) and (13) are stable in the sense of definition (i) if either of them is, and both (1) and (13) are stable in the sense of definition (ii) if either of them is. Both of these criteria (which are quite independent, since (3) does not imply (2)) are contained, as corollaries, in known[5] asymptotic correspondences between the solutions of (1) and (13), when (14) is satisfied. The situation is changed if (i) or (ii) is replaced by (iii). In fact, if $f(t) = t^{-2}$, then (1) is oscillatory (since (9) then holds for an $a > 1/4$), and (14) is satisfied if $g(t) \equiv 0$, but (13) is then non-oscillatory.

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ON PERTURBATION METHODS INVOLVING EXPANSIONS IN TERMS OF A PARAMETER*

By RICHARD BELLMAN (*Rand Corporation*)

Summary. It is shown by means of some examples from the theories of linear algebraic equations, linear integral equations and nonlinear differential equations that the effectiveness of the method of expanding a solution in a power series in terms of a parameter may in many cases be greatly increased by expanding in terms of a suitably chosen function of the parameter. This is particularly the case when the physical setting of the problem allows only positive values of the parameter to enter.

1. Introduction. A standard tool in the theory of functional equations, of both linear and nonlinear character, is the expansion of the solution as a power series in a parameter appearing in the equation, or in the boundary conditions. Some typical examples of equations involving a parameter are

$$
\begin{aligned}
&\text{(a)} && x^2 + x = \lambda, \\
&\text{(b)} && x_i + \lambda \sum_{j=1}^{N} a_{ij}x_j = c_i, \qquad i = 1, 2, \cdots, N, \\
&\text{(c)} && f(x) + \lambda \int_0^1 K(x, y)f(y)\, dy = g(x), \\
&\text{(d)} && x'' + \lambda(x^2 - 1)x' + x = 0 \\
&\text{(e)} && x'' + x + \lambda x^3 = 0.
\end{aligned}
\tag{1.1}
$$

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Two possibilities arise when the formal expansion for the solution is obtained. The expansion may be an asymptotic series, convergent for no non-zero value of the parameter, or it may be a power series possessing a non-zero radius of convergence. We shall be interested here only in the case where there is a finite radius of convergence, although the idea we present is equally applicable to the case of an infinite radius of convergence, in which case we may wish to speed up the convergence of the series, or to the case of an asymptotic series, in which case we may wish to improve the best possible approximation.

As we know, the radius of convergence of a power series is determined by the distance from the origin to the nearest singularity of the function. Consequently, even if we are concerned with determining the numerical value of a function in a region in which it has no singularities, we are very often prevented from using the power series for this purpose because of singularities which occur in regions of no interest to us.

Let us give some examples which we shall discuss again below. If λ is small and positive the positive root of (1.1a) has a power series expansion of the form

$$x = \lambda - \lambda^2 + \cdots = -\frac{1}{2} + \frac{(1 + 4\lambda)^{1/2}}{2}. \tag{1.2}$$

This expansion may only be used up to the value $\lambda = 1/4$, because of a singularity at $\lambda = -1/4$.

Similarly, if we consider the vector-matrix form of (1.1b) $x + \lambda Ax = c$, the Liouville-Neumann solution

$$x = c - \lambda Ac + \cdots \tag{1.3}$$

will have a finite radius of convergence determined by the location of the roots of the determinantal equation $| I + \lambda A | = 0$. If we take A to be a positive definite matrix, and λ a positive number, there will always be a unique solution of (1.1b) for λ sufficiently large. Nevertheless, we cannot use the power series in (1.3) to determine the solution directly.

The same remarks may be made concerning the linear integral equation in (1.1c).

Turning to the differential equations above, the first of which is the famed equation of Van der Pol, and the second the normalized equation of a nonlinear spring, it is known that both equations possess periodic solutions for all positive values of λ, and that these periodic solutions may be represented as power series in λ, for small values of λ, heuristically by means of Lindstedt technique, or rigorously by means of the small-parameter Poincaré method, see Stoker, [5], or Lefschetz, [2].

These power series will not converge for all values of λ. In the Van der Pol case, this is due to the fact that the equation does not possess a periodic solution for large negative values of λ. In the case of the nonlinear spring, this is due to the existence of other singularities of the function.

A question of some importance, from both the theoretical and computational point of view, is whether or not it is possible to obtain expansions which will be valid for all positive values of λ, and, if not that, then at least valid for larger values of the positive parameter than are permitted by the power series expansion.

We shall see that this is rigorously possible in the algebraic cases, and in the case of the linear integral equation, where the analytic character of the solution is readily obtained, and plausibly possible in the case of the Van der Pol and nonlinear spring equations, where the analytic character of the solutions is less known. Actually, since the nonlinear spring equation may be solved explicitly in terms of elliptic functions, an exact investigation would not be too difficult in this case. However, we shall not include it here, since we are primarily interested in presenting and illustrating the basic principle in as simple and direct terms as possible.

The fundamental idea is that of expanding the solution as a power series in another variable ρ which is itself a power series in λ, $\rho = \phi(\lambda)$.

We are essentially, then, presenting a method of analytic continuation. Many examples of this exist in the literature. The classical treatment of the hypergeometric equation exemplifies the general technique, and a particular version will be found in Shohat's generally overlooked paper [4] on the Van der Pol equation. Unfortunately, the idea is a bit obscured by the introduction and subsequent elimination of a fictitious parameter. For a closely related discussion of summability properties of the Liouville-Neumann series obtained from the linear integral equation in (1.1d) see [1].

It is clear that it is easy to multiply examples of physical problems where this technique may be of use. An example of particular importance, which we shall treat elsewhere, is the theory of shock waves of varying strength. As the strength increases, the ordinary perturbation techniques may be expected to be less and less effective. Here the physical background restricts us in a very natural way to the consideration of positive values.

In passing, let us note that the Lighthill technique, see [3], may also be profitably modified in this manner in many cases. We shall discuss this topic in further detail at a subsequent time.

2. A simple algebraic example. Let us begin with an example which, if very simple, has the merit of illustrating the idea very clearly. Consider the problem of finding a power series development for the positive root of $x^2 + x = \lambda$, where $\lambda > 0$. Since we have an explicit formula for the solution in this case, we know that the power series for x in terms of λ, $x = \lambda - \lambda^2 + \cdots$, converges only for $\lambda < 1/4$. Let us perform the change of variable, $\rho = \lambda/(1 + 4\lambda)$. Then $1 + 4\lambda = 1/(1 - 4\rho)$ and

$$x = -\tfrac{1}{2} + \tfrac{1}{2}(1 - 4\rho)^{-1/2} = \rho + 3\rho^2 + \cdots . \tag{2.1}$$

The radius of convergence in the ρ-plane is 1/4. Hence the series converges as long as $|\lambda/(1 + 4\lambda)| < 1/4$, which means that it converges for all positive λ.

The success of the method in this case is due to the fact that we know so much about the analytic character of x as a function of λ. Suppose that we had merely set ρ equal to $\lambda/(1 + \lambda)$. Then

$$x = -\frac{1}{2} + \frac{1}{2}\left(\frac{1 + 3\rho}{1 - \rho}\right)^{1/2} \tag{2.2}$$

which means that the radius of convergence in the ρ-plane is 1/3. Hence as long as $|\lambda/(1 + \lambda)| < 1/3$ the expansion for x in terms of ρ will converge. Since $1/4/(1 + 1/4) = 1/5 < 1/3$, we see that the next expansion will allow larger positive values of λ than before.

3. The solution of $x + \lambda Ax = c$. Let us now turn to a discussion of the solution of the system in (1.1b). The following remarks will apply equally well to the solution of a linear integral equation, such as one in (1.1c), and indeed are abstractly identical if we regard A as a positive operator.

The Liouville-Neumann solution of $x + \lambda Ax = c$,

$$x = c - \lambda Ac + \lambda^2 A^2 c - \cdots , \tag{3.1}$$

obtained by iteration, converges within the circle, $|\lambda| < 1/|\lambda_M|$ where λ_M is the, or a, characteristic root of largest absolute value.

If we take A to be a positive matrix, in the sense below*, the classical theorem of Perron asserts that there is a unique root of largest absolute value which is positive and simple. By a change of variable we may consider this root to be unity.

To take advantage of the positivity of A, and the fact that at the moment we are interested only in positive values of λ, let us make the change of variable, $\rho = \lambda/(1 + \lambda)$, or $\lambda = \rho/(1 - \rho)$. The equation for x becomes

$$x(1 - \rho) + \rho Ax = c - c\rho \tag{3.2}$$

with the power series solution

$$x = c - Ac\rho + \sum_{n=2}^{\infty} \rho^n (A - I)^n (-Ac). \tag{3.3}$$

Since the characteristic roots of $A - I$ are $\lambda_k - 1$, $k = 1, 2, \cdots$, where λ_k are the roots of A, we see that the absolute value of any root of $A - I$ is less than 2. Hence the radius of convergence of the series in (3.3), considered as a series in ρ is greater than 1/2. Since $\lambda/(1 + \lambda) = 1/2$ at $\lambda = 1$, we see that in any case we have enlarged the set of positive values of λ which may be used for the series expansion in ρ.

If, in addition to A being positive, it is also positive definite, then all of the characteristic roots of A will be positive, which means that $0 \le 1 - \lambda_k < 1$, for all k. Hence the series in (3.3) will converge for $|\rho| < 1$, or $|\lambda/(1 + \lambda)| < 1$, which includes all positive values of λ.

In general, we would like to have the ratio of the root of largest absolute value to the root of next largest absolute value as large as possible, in order to speed up convergence. We can accomplish this by iterating in the usual fashion a finite number of times before introducing the above change of variable. For example, iterating twice, we derive

$$x = c - \lambda Ac + \lambda^2 A^2 c - \lambda^3 A^3 x \tag{3.4}$$

or

$$x + \lambda^3 A^3 x = c - \lambda Ac + \lambda^2 A^2 c, \tag{3.5}$$

where we now consider λ^3 to be the new parameter.

4. The nonlinear spring. Let us now turn to the discussion of nonlinear differential equations, considering as a first example the equation

$$x'' + x + \lambda x^3 = 1, \qquad x(0) = 1, \qquad x'(0) = 0. \tag{4.1}$$

Following the ideas of Lindstedt, see [2], [4], [5], to obtain a series expansion for the periodic solution with the above normalized initial conditions, it is necessary to take account of the fact that nonlinearity affects not only the amplitude, but also the frequency. This corresponds to the fact that while linear equations have fixed singularities, nonlinear equations in general have movable singularities, see [3].

Hence we make an initial change of variable, $s = tv$, where v is a parameter depending

*A matrix A is called positive if $a_{ij} > 0$, $(i, j = 1, 2, \cdots, n)$.

upon λ. The new equation is

$$v^2x'' + x + \lambda x^3 = 0, \tag{4.2}$$

which we write in the form

$$(\lambda v)^2x'' + \lambda^2 x + \lambda^3 x^3 = 0. \tag{4.3}$$

Let us set $\rho = \lambda/(1 + \lambda)$, $\lambda = \rho/(1 - \rho)$, and

$$v\lambda = \rho + c_2\rho^2 + c_3\rho^3 + \cdots \tag{4.4}$$

since we want v to be 1 when $\lambda = 0$. We also set

$$x = \cos s + \rho u_1(s) + \rho^2 u_2(s) + \cdots \tag{4.5}$$

and substitute in (4.1). The conditions that u_1, u_2, and so on, be periodic functions of s with period 2π will determine the coefficients in the expansion in (4.4), as in the usual application of the Lindstedt method. The initial conditions, $u_i(0) = u_i'(0) = 0$, will then determine the function $u_i(s)$ for $i = 1, 2, 3, \cdots$.

It is useful to observe that the first approximations obtained from this method and the usual method will agree up to first degree terms in λ. Consequently, a great deal of arithmetic work can be saved by using previous results. Furthermore, where previous perturbation techniques have carried the approximation to higher terms, these series can be transformed to yield the series above.

The calculations indicated above are straightforward, and as always laborious. Computations very kindly performed by George Waters yield the following values for the coefficients in (4.4),

$$\begin{aligned} c_2 &= 1.37500 \\ c_3 &= 1.66797 \\ c_4 &= 1.91845 \\ c_5 &= 2.14083 \end{aligned} \tag{4.6}$$

These coefficients yield a value of 1.24+ for v when $\lambda = 1$, as compared with $v = 1.28$ obtained from a REAC computation.

Let us observe in passing that when the coefficients increase as gradually as those in (4.6), a crude extrapolation will yield values of c_6 and c_7 which will considerably decrease the error made in breaking off the series after five terms. Furthermore, if we are interested in the value $\lambda = 1$, it is probably better to extrapolate the values of $c_n(1/2)^n$ based upon the known values $n = 2, 3, 4, 5$.

5. The Van der Pol equation. Shohat expansion. Turning to the Van der Pol equation, the same change of variable as before yields the equation

$$(\lambda v)^2x'' + (\lambda v)\lambda^2(x^2 - 1)x' + \lambda^2 x = 0. \tag{5.1}$$

Setting

$$\begin{aligned} \lambda &= \rho/(1 - \rho) = \rho + \rho^2 + \rho^3 + \cdots, \\ \lambda v &= \rho + c_2\rho^2 + c_3\rho^3 + c_4\rho^4 + \cdots, \\ x &= \cos s + u_1(s) + u_2(s) + \cdots, \end{aligned} \tag{5.2}$$

the Lindstedt procedure furnishes the following values

$$c_2 = 1, \qquad c_3 = 15/16, \qquad c_4 = 13/16.$$

From the following table, reproduced from Shohat's paper cited above, it is tempting to conjecture that the Shohat series converges for all values of λ which are positive. If so, the series should be more widely known.

λ	v computed using (5.2)	v (Van der Pol)
.33	.98	.99
1.0	.93	.90
2.0	.77	.78
8.0	.35	.39
10.0	.30	.31

The Van der Pol values were obtained using graphical techniques.

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ON MIDDLETON'S PAPER "SOME GENERAL RESULTS IN THE THEORY OF NOISE THROUGH NON-LINEAR DEVICES"*

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As one of the central results of the title paper [1], Middleton obtained $R_l(t)$, the correlation function for the lth zone, as a function of the input correlation function r_0 in the case of the νth law half-wave rectification of narrow-band normal noise (see, e.g., his equations (7.14) and (7.15)). Unless one resorts to series evaluations, his formulas are not particularly suited for numerical computation as they stand, involving as they do hypergeometric functions which are not well tabulated. For purposes of calculation, then, a reduction of the hypergeometric functions occurring in the formulas to tabulated functions must ordinarily be effected, usually by applying the recursion relations among contiguous hypergeometric functions due to Gauss.

When this reduction is accomplished, the hypergeometric functions in Middleton's formulas are seen to be either polynomials in r_0^2 or combinations of complete elliptic integrals of the first and second kind, provided ν is an integer (see, e.g., Middleton's equations (7.16) and (7.17)). These polynomials and combinations of elliptic integrals turn out to be, in every case so far examined, special cases of "Bennett functions" recently tabulated by the author and his colleagues [2, 3]. In the present note expressions

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