have studied the oscillations in an electric circuit containing iron, where the nonlinearity is essentially of the seventh degree. The damping is reduced by an electronic device, so it has been possible to obtain all subharmonics from the second to the ninth order. For small values of ϵ a good agreement with this theory is obtained. The band character and the existence of an upper limit for k has been clearly demonstrated. The subharmonic state was excited by a voltage pulse of the same magnitude as b.

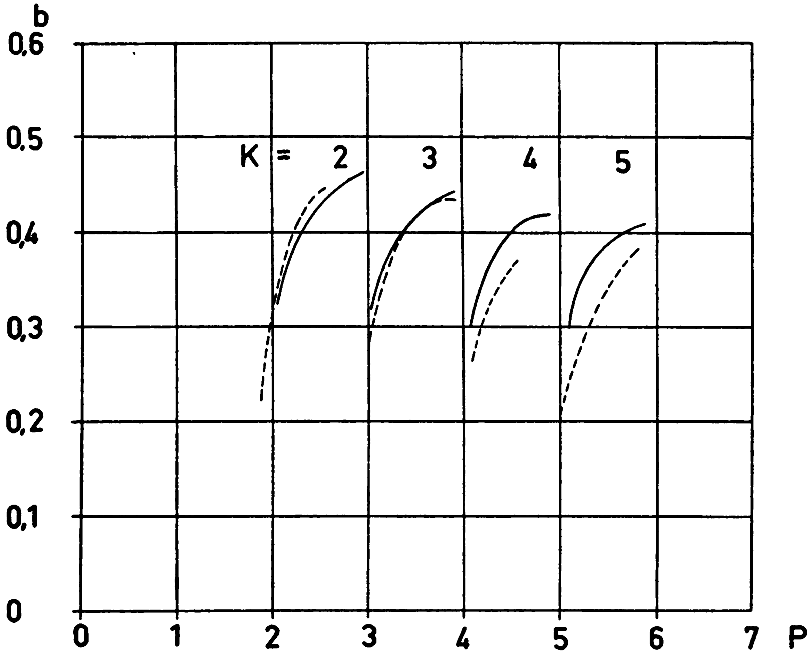

FIG. 5. Experimental curves (dotted) according to K. Göransson and L. Hansson and theoretical values from Eq. (13).

ON THE STABILITY OF THE AXIALLY SYMMETRIC LAMINAR JET*

BY H. G. LEW (*The Pennsylvania State University*)

1. The stability of the axially symmetric laminar jet will be investigated herein subject to rotationally symmetric disturbances. We suppose the laminar jet to be issuing from a small hole with the motion symmetrical about the z axis which is aligned with the jet. The angular position of any point is given by the angle φ measured from the positive x-axis, and r the perpendicular distance from the axis. Thus a set of cylindrical polar coordinates is used.

The steady state flow can be solved in closed form and is given in Goldstein [Ref. 1].

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The distribution of the velocity in the direction of the jet axis is

$$W = \frac{3}{8}\frac{M}{\pi\mu}\frac{1}{z}\frac{1}{(1 + \frac{1}{4}\xi^2)^2}, \tag{1}$$

where M is the rate of momentum flowing across a section of the jet and is a constant, and ξ is defined by

$$\xi = \frac{1}{4\nu}\left(\frac{3M}{\pi\rho}\right)^{1/2}\frac{r}{z}.$$

If we select characteristic length and velocity as

$$L = \left(\frac{M}{\mu\nu z^2}\right)^{-1/2} \tag{2}$$

$$W_0 = \frac{3}{8}\cdot\frac{M}{\pi\mu}\frac{1}{z}$$

respectively, then the velocity W from (1) becomes:

$$\frac{W}{W_0} = \frac{1}{[1 + b(r/L)^2]^2}, \tag{3}$$

where $b = 3/64\pi$.

We consider now the stability of the axially symmetric jet with regard to small rotationally symmetric disturbances of the exponential type. The disturbances are considered small compared with steady-state flow and all non-linear terms of the disturbances are neglected. Moreover, the steady-state flow is to be of the boundary layer type. Thus, we have for the velocities and pressure

$$\begin{aligned} w^* &= W(r) + w(r) \exp [i\alpha(z - ct)], \\ u^* &= \qquad\quad u(r) \exp [i\alpha(z - ct)], \\ v^* &= \qquad\quad v(r) \exp [i\alpha(z - ct)], \\ p^* &= P + \quad p(r) \exp [i\alpha(z - ct)], \end{aligned} \tag{4}$$

where W, P are the steady-state quantities, the others are the disturbances, and α(real) the wave number, and $c = c_r + ic_i$ with c_r the wave velocity and c_i the damping or amplification factor. If Eqs. (4) are inserted into the equations of motion [Ref. 1], neglecting non-linear terms of the disturbances, and also assuming that the steady state flow is of the boundary layer type, we obtain the following disturbance equations:

$$u i\alpha(W - c) = -\frac{1}{\rho}p' + \nu\left(u'' + \frac{1}{r}u' - u\alpha^2 - \frac{u}{r^2}\right), \qquad \text{(a)}$$

$$i\alpha(W - c)v = \nu\left(v'' + \frac{1}{r}v' - v\alpha^2 - \frac{v}{r^2}\right), \qquad \text{(b)}$$

$$i\alpha(W - c)w + uW' = -\frac{1}{\rho}pi\alpha + \nu\left(w'' + \frac{w'}{r} - w\alpha^2\right), \qquad \text{(c)}$$

$$(ru)' + wri\alpha = 0, \qquad \text{(d)}$$

(5)

where the primes denote differentiations with respect to r. The boundary conditions are that the disturbances u, v, w, and p are bounded and that they vanish at infinity. Moreover, all integrals over $(0, \infty)$ are to be convergent.

It is noted that Eq. (5b) is uncoupled from the other equations (5a,c,d). Thus, all discussion of stability with regard to the v-component may be obtained from Equation (5b). The steady-state flow may be shown to be stable to the v-disturbance component in the following way. Following Synge [Ref. 2], we multiply Eq. (5b) by $(r\bar{v})$ (the bar denotes the complex conjugate) and integrate over the interval $(0, \infty)$. Thus we obtain:

$$i\alpha \int_0^\infty (W - c)\cdot v\bar{v}r\, dr = \nu \int_0^\infty \left[v''\bar{v}r + \bar{v}v' - r\bar{v}v\alpha^2 - \frac{v\bar{v}}{r} \right] dr. \tag{6}$$

If we integrate the first term of Eq. (6) by parts, assert the boundary condition $v(+\infty) = 0$, and separate real and imaginary parts, we obtain

$$c_r = \frac{\displaystyle\int_0^\infty Wrv\bar{v}\, dr}{\displaystyle\int_0^\infty rv\bar{v}\cdot dr}$$

and

$$c_i = -\frac{\nu}{\alpha} \int_0^\infty \left[v'\bar{v}' + \bar{v}v\alpha^2 + \frac{v\bar{v}}{r} \right] dr,$$

assuming that v vanishes at least by $r^{-1/2}$ at infinity. Thus c_i is always negative and therefore the jet is stable to the v disturbance. Hence, in this case of rotationally symmetric disturbances, it is sufficient to investigate the case of axially symmetric disturbances.

2. We consider the inviscid case now for the u, w and p components. Setting $\nu = 0$ in Eqs. (5a) and (5c), eliminating the p term by the usual differentiation and subtraction processes, and utilizing the continuity Eq. (5d), we obtain one equation for the u component

$$(W - c)\left[u'' - \alpha^2 u + \left(\frac{u}{r}\right)' \right] - u\left(W'' - \frac{W'}{r} \right) = 0. \tag{7}$$

We may non-dimensionalize Eq. (7) by using the characteristic length and velocity given by Eq. (2); for example, the non-dimensional disturbance velocity and coordinate r are u/W_0, r/L, etc. The resulting equation has the same form as that of Eq. (7), and hereafter, we note that Eq. (7) and all subsequent equations in this section, have been non-dimensionalized and no new equations will be introduced. Here we note the values of the steady state velocity and its derivatives for future use. They are:

$$\begin{aligned} W &= \frac{1}{(1 + br^2)^2}, \\ W'' - \frac{W'}{r} &= \frac{24b^2r^2}{(1 + br^2)^4}. \end{aligned} \tag{8}$$

We can show that for the axially symmetric jet given by Eqs. (8) there exist no self-excited disturbances and no damped disturbances when viscosity is neglected. The proof is as follows:

Consider a transformation for $u(r)$ of the form:

$$u(r) = r^{-1/2}g(r). \tag{9}$$

Insertion of Eq. (9) in the disturbance Eq. (7) leads to

$$g'' + \left(-\frac{3}{4r^2} - \alpha^2\right)g - \frac{W'' - r^{-1}W'(r)}{W - c}\cdot g = 0. \tag{10}$$

Let

$$\begin{aligned} L[g] &= g'' - \left(\frac{3}{4r^2} + \alpha^2\right)g, \\ f(r) &= -\frac{W'' - r^{-1}W'(r)}{W - c}, \end{aligned} \tag{11}$$

then Eq. (10) can be written as:

$$L[g(r)] + \cdot f(r)\cdot g(r) = 0. \tag{12}$$

We assume a non-neutral oscillation so that there is a solution g with c not real. We multiply Eq. (12) by the complex conjugate $\bar{g}$ and integrate over $(0, \infty)$ and subtract its conjugate. Thus

$$\int_0^\infty \{\bar{g}L[g] - gL[\bar{g}]\}\, dr + \int_0^\infty \bar{g}g(f - \bar{f})\, dr = 0 \tag{13}$$

and since

$$\int_0^\infty \{\bar{g}L[g] - gL[\bar{g}]\}\, dr = 0, \tag{14}$$

then

$$\int_0^\infty \bar{g}g(f - \bar{f})\, dr = -2ic_i \int_0^\infty \bar{g}g \frac{W'' - r^{-1}W'(r)}{| W - c |^2}\, dr = 0. \tag{15}$$

However, since Eq. (8) shows that $(W'' - W'/r)$ is always positive in $(0, \infty)$, Eq. (15) is satisfied only if $c_i = 0$. Hence there are no self-excited or damped disturbances if viscosity is neglected, and therefore only neutral disturbances need be considered. This result is interesting since any section taken of the jet containing the jet axis leads to a velocity distribution with two inflection points for which non-neutral oscillation may occur if that profile is a two-dimensional one. Of course, here we do not have a true inflection point in that sense since the flow is three dimensional, and rotational symmetric disturbances are considered only.

We shall consider now the case of the neutral mode. Suppose there is a point r_0 such that $[W''(r_0) - W'(r_0)/r_0] = 0$ in the interval $(0, \infty)$, then r_0 is either 0 or ∞ by Eq. (8). At $r_0 = 0$ and ∞ we have, respectively,

$$W(0) = 1 \qquad \text{and} \qquad W(\infty) = 0.$$

Now consider the differential equation

$$g'' + \lambda g + Kg = 0, \tag{16}$$

where

$$K = -\frac{3}{4r^2} - \frac{W''(r) - r^{-1}W'(r)}{W(r) - W(r_0)}$$

and

$$W(r_0) = 0 \quad \text{or} \quad 1.$$

Equation (16) is equivalent to a variational problem for the first eigenvalue of

$$\lambda = \min \frac{\int_0^\infty (g'^2 - Kg^2)\, dr}{\int_0^\infty g^2\, dr}. \tag{17}$$

We must show that the ratio in Eq. (17) is negative for some function g satisfying the boundary condition. Then $\lambda < 0$ and the minimum will then be below this value. For the case of $W(r_0) = 0$, K will always be negative and therefore all characteristic numbers λ are positive and no neutral oscillation occurs. In the case of $W(r_0) = 1$ at $r_0 = 0$, K may be positive, and there may exist a function such that the ratio in Eq. (17) is negative. Numerical calculations for this case are given in [Ref. 3].

It is interesting to note that the term

$$\frac{1}{r}\left(W'' - \frac{W'}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\frac{dW}{dr}\right)$$

is the gradient of $-\eta/r = r^{-1}W'(r)$ where η is the vorticity of the basic flow in accordance with the parallel flow assumption. Thus, the condition for neutral oscillation corresponds to the vanishing of this gradient at some point. This is somewhat analogous to the two-dimensional case discussed by Lin in [Ref. 4] where a physical interpretation of this condition for stability is given. However, here the important quantity is η/r and not the vorticity alone.

The extension of these discussions to three-dimensional disturbances should allow for the dependency of the disturbances on the angular variable φ.

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ON SOURCE AND VORTEX OF FLUCTUATING STRENGTH TRAVELLING BENEATH A FREE SURFACE*

By H. S. TAN (*University of Notre Dame*)

In the coordinate system which moves with the travelling source or vortex at constant forward speed c, and under the hypothesis that the resulting fluid motion is irrotational, one can define a disturbance velocity potential $\Phi(x, y, t)$ for the two dimensional fluid motion through the differential equation

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