ON DIFFUSIVE CONVECTION IN TUBES*

By G. F. CARRIER (*Harvard University*)

1. Introduction. There is a group of experiments (as exemplified by the configuration of Fig. 1) in which one determines the time dependent solute content of a

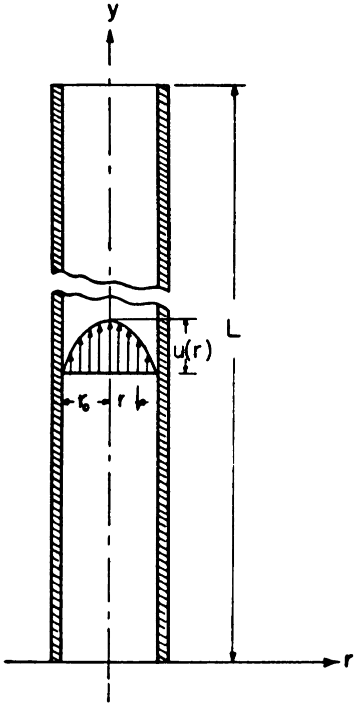

FIG. 1. The solute is convected into the tube at $y = 0$, the entrance condition being $s = s_0 + \cos \Omega t$.

fluid by continuously collecting the fluid through a long tube (length/radius $\ll 1$) and appropriately testing samples so collected for solute content. The interpretative question which arises is: "What is the relationship between the solute concentration at the source and that at the tube exit?" The answer to this question is found here for one range[1] of values of the important parameters. The results are of interest in that the following unexpected[2] conclusion is reached. The attenuation in the solute concentration from source to tube exit can be smaller for a solute with a high diffusivity than for one with a smaller (or zero) diffusivity. That is, the diffusivity improves the response of the "instrument" over the response that would be observed were the process purely convective.

2. Analysis of the problem. The solute concentration[3] of the fluid in the tube is

*Received April 23, 1955.

[1]This range is encountered in experimentation concerning the salinity of ground water in permeable islands.

[2]I.e., unexpected by the author.

[3]Actually s is the difference between the gross fluctuating concentration and the average concentration.

denoted by $s(r, y, t)$ and its distribution is governed by convection and diffusion according to the law

$$s_t + u_0(1 - r^2)s_y = \nu\Delta s. \tag{2.1}$$

We assume here that the flow rate is not time dependent and that the Reynolds number is low (in the problem of footnote 1, $\rho u_0 r_0/\mu = \text{Re} = 0\,[100]$) so that the flow is laminar. In the foregoing ν is the diffusivity, u_0 the maximum velocity, y and $r\,r_0$ are the cylindrical coordinates[4] as indicated in Fig. 1, and Δ is the Laplace operator. The initial condition states that[5]

$$s(r, 0, t) = \exp(i\Omega t).$$

We wish to find the ratio of the solute discharge at $y = L$ to the intake at $y = 0$. That is, we wish to compute

$$R = \int_0^1 r(1 - r^2)s(r, L, t)\,dr \Big/ \int_0^1 r(1 - r^2)s(r, 0, t)\,dr. \tag{2.2}$$

It is clear that $s(r, y, t)$ can be represented in the form

$$s(r, y, t) = \sum_{n=0}^{\infty} F_n f_n(r) \exp(i\Omega t - \beta_n y) \tag{2.3}$$

and that this will be an "efficient" representation if $\text{Re}(\beta_n)$ increases so rapidly with n that $|\exp(-\beta_N L)| \ll |\exp(-\beta_0 L)|$ for moderately small N. As we shall see, there is a large range of the parameter $\omega = \Omega r_0^2/\nu$ for which the foregoing is true and the use of Eq. (2.3) is therefore to be adopted. If, then, we use Eq. (2.3) and introduce the notation $\alpha_n = r_0^2\, u_0\beta_n/\nu$, Eq. (2.1) becomes

$$r^{-1}(rf_n')' - (i\omega - \alpha_n[1 - r^2] - \epsilon\alpha_n^2)f_n = 0, \tag{2.4}$$

where $\epsilon = \nu^2/u_0^2r_0^2$. It can be anticipated that the results of interest (i.e., the not too strongly attenuated signal cases) will occur for those situations where $\epsilon \ll 1$ and it can also be anticipated that α_0 will be of order ω with interest centering on the case $\omega = 0(1)$. For such cases, it is clear that the term in α_n^2 is of negligible importance and may be disregarded. (In the ground-water problem alluded to previously ω was about 2 and ϵ was 10^{-9}.) On this basis we omit the ϵ term with the understanding that its importance for any case can be checked when the answers are in by computing $\epsilon\alpha_n^2/[i\omega - \alpha_n(1 - r^2)]$. Consequently, we deal with the equation

$$r^{-1}(rf')' - (i\omega - \alpha[1 - r^2])f = 0, \tag{2.5}$$

where we have omitted subscripts. The boundary conditions require that $f'(0) = f'(1) = 0$ since there must be no diffusion across the tube wall and since the origin is not a singular point.

One way to obtain an accurate representation of the solutions of Eq. (2.5) for, say, $\omega < 5$, is to expand f in the form

$$f(r) = \sum_{p=0}^{\infty} \varphi_p(r)\sigma^p,$$

with

$$\alpha(\sigma) = \sum_{p=0}^{\infty} a_p\sigma^p, \tag{2.6}$$

[4] r_0 is the radius of the tube and r the dimensionless coordinate.

[5] We omit writing Re (real part of) for brevity.

where $\sigma = i\omega$. We can anticipate that the "lowest" eigenfunction will have associated with it an $\alpha(\sigma)$ such that $\alpha(0) = 0$. (Note that for $\sigma = 0$; $f \equiv 1$ and $\alpha = 0$). Thus, the successive φ_p obey the equations

$$r^{-1}(r\varphi_p')' = [\varphi_{p-1} - (a_p\varphi_0 + a_{p-1}\varphi_1 + \cdots + a_1\varphi_{p-1})(1 - r^2)]. \tag{2.7}$$

Equations (2.7) can be integrated successively starting with $p = 0$ and proceeding monotonically to larger p. In each integration the constants of integration can be arbitrarily taken to be zero. (Other choices which do not violate $\varphi_p'(0) = 0$ merely multiply the final result by a numerical factor.) The number a_p is chosen so that the boundary condition $\varphi_p'(1) = 0$ is satisfied. This procedure leads to the result

$$f_0 = 1 + \sigma(x^4/8 - x^2/4) + \sigma^2(x^2/96 + 5x^4/384 - 5x^6/288 + x^8/256) + \cdots \tag{2.8}$$

$$\alpha_0 = 2\sigma - \sigma^2/24 + \sigma^3/960 + \cdots$$

It can be seen that the convergence is excellent for moderate values of $|\sigma|$.

An alternative method of deducing Eq. (2.8) is to form the variational problem

$$\delta \int_0^1 \{f_r^2 + [\sigma - \alpha(1 - r^2)]f^2\}r\, dr = 0. \tag{2.9}$$

If we use the usual Rayleigh-Ritz procedure taking f to be a polynomial $\sum a_n r^{2n}$ for $0 \leq n \leq 3$, and impose the proper boundary condition ($f_r(1) = 0$), we obtain a characteristic equation which is cubic in α and which, in particular, gives $\alpha_0 = 2\sigma - \sigma^2/24 + 0(\sigma^3)$ again. It also gives $\alpha_1 \simeq 26 + 0(\sigma)$ and $\alpha_2 \simeq 160$. The value for α_2 is certainly not to be trusted, but the value for α_1 is an excellent estimate for moderate σ (say $|\sigma| < 5$). If we adopt 26 as the order of magnitude of α_1, then, we note that $(\beta_1 L - \beta_0 L)$, which is a measure of the relative importance of f_0 and f_1, will obey the following inequality: Re $(\beta_1 - \beta_0)L < A$ when $\omega^2 < 24(26 - A\, r_0^2 u_0/\nu L)$. This defines, in a rough way, the range of utility of this analysis. The foregoing variational procedure also gives an estimate for the eigenfunction $f_1(r)$ which we shall not bother to record.

The solute distribution $s(r, y, t)$ can now be determined by noting that the "Fourier coefficients," F_n, of Eq. (2.3) are those associated with the expansion

$$\sum_{n=0}^{\infty} F_n f_n(r) = 1.$$

The orthogonality relation among the $f_n(r)$ is

$$\int_0^1 f_n(r) f_m(r)(1 - r^2)r\, dr = 0, \qquad \text{for} \qquad m \neq n$$

so that F_n becomes

$$F_n = \int_0^1 f_n(r)(r - r^3)\, dr \Big/ \int_0^1 f_n^2(r)(r - r^3)\, dr. \tag{2.10}$$

For $|\sigma| < 5$ we again can show that $|F_0|$ is close to unity (we shall give an example soon) and[6] $|F_1| < .01$. It is clear that, for these values of σ, the ratio R defined by Eq.

[6]This is associated with the fact that, for $|\sigma|$ small, $f_0(r)$ is close to unity and all other $f_n(r)$ are orthogonal to $f_0(r)$. Hence $f_n(r)$ is nearly orthogonal to the function $h(r) \equiv 1$.

(2.2) is dominated by errors of order $10^{-4} \exp - (\beta_1' - \beta_0')L$ (where β' indicated Re β) by the f_0 contribution and is therefore given by

$$R = \left\{ 4\left[\int_0^1 f_0(r)(r - r^3)\, dr \right]^2 \Big/ \int_0^1 f_0^2(r)(r - r^3)\, dr \right\} \exp(-\alpha_0 L). \tag{2.11}$$

The evaluation of these integrals leads to the result

$$R = \frac{1 - \sigma/8 + 47\sigma^2/3840 + \cdots}{1 - \sigma/8 + 37\sigma^2/3840 + \cdots} \exp(-\alpha_0 L). \tag{2.12}$$

For the range of σ we have mentioned, R is very close to unity. In fact, for one case of interest, $\Omega = 2\pi/\text{day}$, $\nu = 1\ \text{cm}^2/\text{day}$, $u_0 = 10^5\ \text{cm/day}$, $L = 6.10^3$ cm, $r_0^2 = .4\ \text{cm}^2$, so that $\sigma = i\omega = 2.5i$, $\alpha_0^2 = 5i - .252$, Re $(\beta_0 L) = .038$, Re $(\beta_1 L) \simeq 4$, and it is clear that the second mode contribution is only $10^{-4} \exp(-4) = 0(10^{-6})$ times that of the first mode. Thus

$$R = .984 \exp(-.038) = .946.$$

This number is so close to unity that one wonders what it might have been were there no diffusion. The solution of Eq. (2.1) for $\nu = 0$ (suppressing, necessarily, the condition $s_r(1, y, t) = 0$) is

$$s = \exp i\Omega\{t - y/u_0(1 - r^2)\}. \tag{2.13}$$

The efflux ratio, R_0 (meaning R for $\nu = 0$) is given by

$$R_0 = 4 \left| \int_0^1 r(1 - r^2) \exp\{-i\Omega L/u_0(1 - r^2)\}\, dr \right|. \tag{2.14}$$

This integral may be evaluated to give

$$\begin{aligned} R_0 &= \left| (1 - m)e^{-m} + m^2 \int_m^\infty u^{-1} \exp(-u)\, du \right| \\ &= |\, (1 - m)e^{-m} - m^2 E_i(-m)\, |, \end{aligned} \tag{2.15}$$

where $m = i\Omega L/u_0$. In the foregoing example, $L/u_0 = 3/8$ and $R_0 = .91$. Thus, the attenuation is less for the diffusive model than for that with $\nu = 0$. However the asymptotic behavior of R_0 with regard to m for the non-diffusing case is given by $R_0 \sim O(m^{-1})$. This implies that for sufficiently large L, the diffusive attenuation would be much greater than that for the non-diffusive case since the diffusive attenuation is exponential in, say, y. Nevertheless, it is curious that, in some low attenuation cases, the diffusive transport should decay less rapidly than the analogous non-diffusive transport.

Aside from the foregoing result, one should also note that one can use the sampling technique associated with the foregoing over quite a large range of conditions and obtain accurate information with essentially no complicated data reduction. It appears, in fact, that the interpretation is so simple for these cases that a deliberate increase in tube length would sometimes be profitable in the experimental design.

3. Comparison with results of G. I. Taylor. After the foregoing work was completed the author's attention was drawn to the analysis of a similar problem [1]. In that paper, the dispersion of particular (non-oscillatory) distributions of solute was studied. In particular, it was noted that if a concentrated sample of solute were placed at $y = 0$ at time zero the dispersion would occur in a manner equivalent to that in a tube with rigid body flow (u independent of r) and effective diffusivity $\nu' = u_0^2 r_0^2/192\nu$. If we identify the results of the analysis of Sec. 2 with those obtained when we write

$$S_t + (u_0/2)S_y = \nu' S_{yy} , \tag{3.1}$$

(both our results and [1] imply that the transport speed is $u_0/2$), we also obtain the above value for ν' provided $\Omega r_0^2/12\nu \ll 1$.

If now we use a Fourier synthesis over Ω(i.e. if we write $S = \int_{-\infty}^{\infty} H(\omega) \sum_n F_n f_n(\omega, r)$ exp $(i\Omega t - \beta_n r)\, d\Omega$) to describe the salinity distribution of Taylor's problem, it is clear that the large time behavior would be governed by the small Ω behavior of the oscillatory solution and, furthermore, the only eigenfunction which would contribute appreciably would be f_0. That is, the solution to the problem where a delta function distribution of solute was introduced at $t = 0$ and $y = 0$ would be of the form

$$S(y, r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_0(r, \Omega) \exp\left(i\Omega t - \beta_0(\Omega) y\right) d\Omega. \tag{3.2}$$

and a steepest descent evaluation of this integral would lead precisely to the result given by Taylor.

The only advantages of this approach over that of [1] are that: (1) it leads to the observation that the equivalent diffusivity concept is valid only when the solute has traveled an appreciable distance down the tube; and (2) the formal procedure used here makes it possible to improve the accuracy of the prediction in the unlikely event that such improved accuracy were required.

In so far as the problem with oscillating salinity intake is concerned, the analysis of [1] would have predicted the response as .962 instead of .946 since the discrepancy between unity and the Fourier coefficient of f_0 was neglected in the analysis of [1].

Bibliography

[1] Sir Geoffrey Taylor, *Dispersion of soluble matter in solvent flowing slowly through a tube*, Proc. Roy. Soc. **219**, 186 (1953)