

THE DIFFRACTION OF A CYLINDRICAL PULSE BY A HALF-PLANE*

BY

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I. Introduction. In the past fifty-eight years, many papers have been written on the theory of diffraction in two dimensions by a wedge. Since the time of Sommerfeld's attack on the half-plane [1]**, several approaches, using various specialized forms of excitation, have been investigated. Sommerfeld's use of multi-valued functions for plane wave excitation has been supplemented by Macdonald's expansion in the appropriate eigenfunctions for the problem [2]. Much of the work that has been done in this field has been for the domain of harmonic time-dependence; in a sense, these results represent complete solutions for the particular form of excitation involved. However, the inverses of these Fourier-transform solutions are rarely (if ever) determined.

Recently, more direct attacks on the time-dependent wave equations have been successful. Keller and Blank [3] have obtained useful results for the scattering of plane waves by wedges and corners. By recognizing the nature of propagation of discontinuities for solutions of hyperbolic partial differential equations, they are able to make an appropriate change of independent variables. A change of dependent variable yields the Laplace equation, which is then solved by conformal mapping. More recently, Kay [4] has achieved a rather general result, in that he is able, at least in principle, to write down the solution for an arbitrary form of excitation. The method is to make a change of independent variables suggested by the work of Keller and Blank. This leads to a non-orthogonal co-ordinate system, and he devises an integral transformation to treat the rather complicated differential equation he obtains upon separation of variables. The transform kernel involves a Whittaker function, and the integration is over a reasonably complicated contour in the complex plane of the separation parameter. This brings up what seems to be the only limitation on his method—the mathematician's ingenuity in carrying out the details of the analysis. The intricacy of the transformation precludes the possibility of obtaining a more general result than he does—viz., the verification of the results of Keller and Blank.

The method of attack that we propose is less sophisticated than the foregoing, but possesses the advantage that we can determine the Green's function for the problem. For simplicity, we restrict ourselves to the case where the wave-function is constrained to vanish on a half-plane, the excitation being a line source parallel to the edge of the half-plane. The analysis is no more difficult for the general case, however, and may even be extended to the diffusion equation.

II. Explicit statement of the problem. We wish to determine a scalar function ϕ such that

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\delta(r - r_0) \delta(\theta - \theta_0) \delta(t - 0^+)}{r_0} \quad (1)$$

subject to the boundary condition

$$\phi = 0 \quad (2)$$

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**Numbers in square brackets refer to the bibliography at the end of this paper.

for $\theta = 0$ and $\theta = 2\pi$ and the initial conditions

$$\phi = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial t} = 0 \quad (3)$$

at

$$t = 0.$$

Figure 1 shows the relationship between the source point (r_0, θ_0) , the observation point

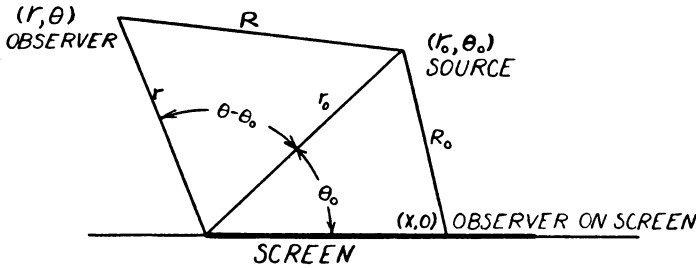


FIG. 1. Relationship between source, observer, and diffracting screen.

(r, θ) , and the diffracting screen $\theta = 0$ (or $\theta = 2\pi$). For convenience, we denote by R the distance from the source point to the observation point:

$$R = [r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)]^{1/2}; \quad (4)$$

when the observation point is on the screen, we call the distance R_0 :

$$R_0 = [x^2 + r_0^2 - 2xr_0 \cos \theta_0]^{1/2}. \quad (5)$$

The problem is reduced to a simpler boundary-value problem at once by writing

$$\phi = \phi^{\text{ino}} + u \quad (6)$$

where [5, p. 332, Eq. 57]

$$\phi^{\text{ino}} = \begin{cases} 0 & t < R/c \\ (1/2\pi)[t^2 - R^2/c^2]^{-1/2} & t > R/c \end{cases} \quad (7)$$

ϕ^{ino} is the free-space Green's function for the two-dimensional wave equation, and satisfies the inhomogeneous wave equation (1) as well as the initial conditions (3). Accordingly,

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad (8)$$

and on the screen ($\theta = 0$ and $\theta = 2\pi$),

$$u = \begin{cases} 0 & t < R_0/c \\ (-1/2\pi)(t^2 - R_0^2/c^2)^{-1/2} & t > R_0/c \end{cases} \quad (9)$$

Also, u satisfies the initial conditions (3). The problem is then to find u .

III. Solution of the problem. We now consider the one-sided Laplace transform of u , which we denote by U . On making use of the initial conditions (3), the partial differential equation transforms to

$$\nabla^2 U - (p^2/c^2)U = 0,$$

and the boundary condition becomes [6, p. 125]

$$U = (-1/2\pi)K_0(pR_0/c)$$

on the screen. The function K_0 is the zero-order Macdonald function [7, p. 78], which is proportional to the zero-order Hankel function of the first kind with imaginary argument.

It is convenient to discuss the function

$$v = U + (1/2\pi)K_0(pr_0/c), \quad (10)$$

which vanishes with r . After multiplying by r^2 , the partial differential equation satisfied by v is:

$$\frac{r^2}{\partial r^2} \frac{\partial^2 v}{\partial r^2} + \frac{r}{\partial r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial \theta^2} - \frac{p^2 r^2}{c^2} v = -\frac{p^2 r^2}{c^2} (1/2\pi)K_0(pr_0/c) \quad (11)$$

and the boundary condition on v is that

$$v = (1/2\pi)[K_0(pr_0/c) - K_0(pR/c)] \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = 2\pi. \quad (12)$$

We now make use of the Grünberg modification [8] of the integral transformation of Kontorowich and Lebedev [9, 10]. This states that if

$$V(s) = \int_0^\infty v(pr/c)K_s(pr/c) dr/r, \quad (13)$$

where K_s is the Macdonald function of order s , then

$$v(pr/c) = (1/\pi i) \int_{-i\infty}^{i\infty} V(s)I_s(pr/c)s ds, \quad (14)$$

where I_s is the modified Bessel function of order s [7, p. 77]. In transforming (11) by means of (13), we integrate by parts several times and make use of the modified Bessel equation satisfied by K_s , obtaining finally:

$$\frac{\partial^2 V}{\partial \theta^2} + s^2 V = -\frac{sK_0(pr_0/c)}{4 \sin(\pi s/2)}. \quad (15)$$

Here we have made use of the formula

$$\int_0^\infty K_s(x)x dx = \frac{\pi s}{2 \sin(\pi s/2)}$$

[7, p. 388, Eq. 8; 6, p. 1]. The boundary condition (12) must also be transformed; at $\theta = 0$ and $\theta = 2\pi$, we have $V = V_0$, where

$$V_0 = (1/2\pi) \int_0^\infty K_s(pr/c)[K_0(pr_0/c) - K_0(pR_0/c)] dr/r. \quad (16)$$

This integral is evaluated in the appendix. We have now reduced the partial differential equation to a particularly simple ordinary differential equation. All that remains is to solve the ordinary differential equation and carry out the two inversions for the two transformations used. The solution to the differential equation is

$$V(s) = A(s) \sin \theta s + B(s) \cos \theta s - \frac{K_0(pr_0/c)}{4s \sin (\pi s/2)}.$$

On applying the boundary condition, we have

$$V(s) = \left[V_0 + \frac{K_0(pr_0/c)}{4s \sin (\pi s/2)} \right] \frac{\cos (\pi - \theta)s}{\cos \pi s} - \frac{K_0(pr_0/c)}{4s \sin (\pi s/2)}.$$

Inserting the expression for V_0 , we have:

$$V(s) = K_*(pr_0/c) \frac{\cos (\pi - \theta_0)s \cos (\pi - \theta)s}{s \sin 2\pi s} - \frac{K_0(pr_0/c)}{4s \sin (\pi s/2)}.$$

Note that V is analytic at $s = 0$. (This is easily seen if we recall that $K_*(z)$ is even in s [3, p. 79, Eq. 8], so that for $s \rightarrow 0$, $K_*(z) = K_0(z) (1 + as^2 + \dots)$.)

The next task is to invert this transform:

$$u = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} \left[K_*(pr_0/c) \frac{\cos (\pi - \theta_0)s \cos (\pi - \theta)s}{s \sin 2\pi s} - \frac{K_0(pr_0/c)}{4s \sin (\pi s/2)} \right] I_*(pr/c)s ds \quad (17)$$

For $r < r_0$, we may use the inversion formula (14) as it stands, computing the integral by closing the contour with a semicircular arc on the right as in Fig. 2, and applying

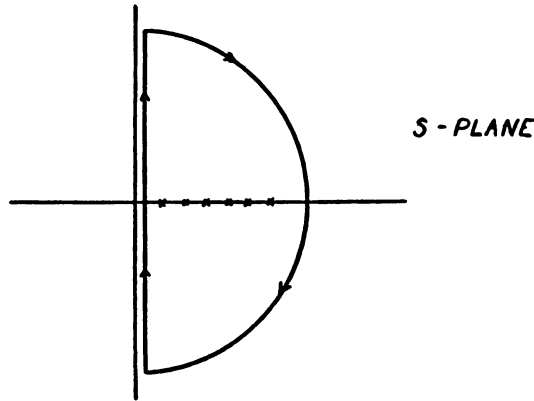


FIG. 2. Contour for inversion of the transform.

the theory of residues. If we choose the radius of the semicircle to be $\frac{1}{4}(2m + 1)$, where m is a positive integer, it is not too hard to show that the integral over the semi-circular arc must vanish as $m \rightarrow \infty$. Thus,

$$v = -2 \sum \text{residues inside the contour} = v_1 + v_2,$$

where

$$v_1 = -2 \sum_{n=1}^{\infty} \text{residues of } \frac{K_0(pr_0/c) I_1(pr/c) \cos(\pi - \theta)s \cos(\pi - \theta_0)s}{\sin 2\pi s}$$

at $s = n/2$;

$$v_2 = -2 \sum_{n=1}^{\infty} \text{residues of } \frac{K_0(pr_0/c) I_1(pr/c)}{4 \sin(\pi s/2)} \quad \text{at } s = 2n.$$

We evaluate v_2 first, since it is the simplest:

$$v_2 = (1/\pi) K_0(pr_0/c) \sum_{n=1}^{\infty} (-1)^n I_{2n}(pr/c).$$

This is easily summed by means of a well-known summation formula for Bessel functions [7, p. 34, p. 77]:

$$v_2 = (1/2\pi) K_0(pr_0/c) - (1/2\pi) K_0(pr_0/c) I_0(pr/c).$$

Note that the first term of v_2 is exactly what we added to U in Eq. (10). Next,

$$v_1 = (-1/\pi) \sum_{n=1}^{\infty} (-1)^n K_{n/2}(pr_0/c) I_{n/2}(pr/c) \cos \frac{1}{2} n(\pi - \theta) \cos \frac{1}{2} n(\pi - \theta_0).$$

After some trigonometric manipulation, we find that

$$\begin{aligned} v = (1/2\pi) K_0(pr_0/c) + (1/\pi) \sum_{n=1, \text{ odd}}^{\infty} K_{n/2}(pr_0/c) I_{n/2}(pr/c) \sin \frac{1}{2} n\theta \sin \frac{1}{2} n\theta_0 \\ - \frac{1}{4\pi} \left\{ K_0(pr_0/c) I_0(pr/c) + 2 \sum_{n=1}^{\infty} K_n(pr_0/c) I_n(pr/c) \cos n(\theta + \theta_0) \right\} \\ - \frac{1}{4\pi} \left\{ K_0(pr_0/c) I_0(pr/c) + 2 \sum_{n=1}^{\infty} K_n(pr_0/c) I_n(pr/c) \cos n(\theta - \theta_0) \right\}. \end{aligned}$$

The bracketed terms may be summed [7, p. 361, Eq. 8], giving finally:

$$\begin{aligned} v = (1/2\pi) K_0(pr_0/c) + (-1/4\pi) [K_0(pR/c) + K_0(pR_1/c)], \\ + (1/\pi) \sum_{n=1, \text{ odd}}^{\infty} K_{n/2}(pr_0/c) I_{n/2}(pr/c) \sin \frac{1}{2} n\theta \sin \frac{1}{2} n\theta_0, \end{aligned}$$

where

$$R_1 = [r^2 + r_0^2 - 2rr_0 \cos(\theta + \theta_0)]^{1/2}; \quad (18)$$

R_1 is the distance of the observation point from the image (with respect to the plane $y = 0$) of the source point. Finally,

$$\begin{aligned} U = (1/\pi) \sum_{n=0}^{\infty} K_{n+1/2}(pr_0/c) I_{n+1/2}(pr/c) \sin \left(n + \frac{1}{2}\right) \theta \sin \left(n + \frac{1}{2}\right) \theta_0 \\ - (1/4\pi) [K_0(pR/c) + K_0(pR_1/c)]. \quad (19) \end{aligned}$$

Equation (19) is valid only if $r < r_0$. If $r > r_0$, it is necessary to revise the form of the inversion integral. This is done by utilizing the defining equation for the Macdonald function [7, p. 78, Eq. 6] and the symmetry of $V(s)$, which is always even. Doing this, Eq. (14) becomes

$$v = (1/\pi i) \int_0^{i\infty} V(s) (-2/\pi) K_0(pr/c) s \sin \pi s \, ds.$$

Now when we insert the expression for $V(s)$ into the inversion formula, the term involving $K_0(pr_0/c)$ will give the same result as before, when we sum over its residues. The other term turns out to be just the integral for V_1 that we evaluated previously, with r and r_0 interchanged. This merely interchanges r and r_0 in the expression (19) for U . Introducing the shorthand notation: $r_> = \max(r, r_0)$, $r_< = \min(r, r_0)$, the two expressions for U may be combined:

$$U = (1/\pi) \sum_{n=0}^{\infty} K_{n+1/2}(pr_>/c) I_{n+1/2}(pr_</c) \sin\left(n + \frac{1}{2}\right)\theta \sin\left(n + \frac{1}{2}\right)\theta_0 \\ - (1/4\pi)[K_0(pR/c) + K_0(pR_1/c)]. \quad (20)$$

The next step is the inversion of U . This is facilitated by use of the integral of Sonine and Gegenbauer [7, p. 367, Eq. 17]. Specialized to our problem, this reads:

$$K_{n+1/2}(pr_>/c) I_{n+1/2}(pr_</c) \\ = \frac{1}{2} (rr_0)^{1/2} \int_0^\pi \frac{\exp[-(p/c)(r^2 + r_0^2 - 2rr_0 \cos \phi)^{1/2}]}{(r^2 + r_0^2 - 2rr_0 \cos \phi)^{1/2}} P_n(\cos \phi) \sin \phi \, d\phi,$$

when P_n denotes the Legendre polynomial of order n . But this last integral is the Laplace transform of

$$\frac{1}{2} (rr_0)^{1/2} \int_0^\pi \frac{\delta[t - (1/c)(r^2 + r_0^2 - 2rr_0 \cos \phi)^{1/2}]}{(r^2 + r_0^2 - 2rr_0 \cos \phi)^{1/2}} P_n(\cos \phi) \sin \phi \, d\phi.$$

Performing the indicated integration, we find that

$$K_{n+1/2}(pr_>/c) I_{n+1/2}(pr_</c)$$

is the Laplace transform of the function

$$\begin{cases} \frac{1}{2} c(rr_0)^{-1/2} P_n \left[\frac{r^2 + r_0^2 - c^2 t^2}{2rr_0} \right] & |r - r_0| < ct < r + r_0 \\ 0 & \text{otherwise.} \end{cases}$$

The remaining terms of (20) are easily inverted [6, p. 125], so that $u = u_1 u_3 - u_2$, where

$$u_1 = \begin{cases} \frac{c}{2\pi(rr_0)^{1/2}} \sum_{n=0}^{\infty} P_n \left[\frac{r^2 + r_0^2 - c^2 t^2}{2rr_0} \right] \sin\left(n + \frac{1}{2}\right)\theta \sin\left(n + \frac{1}{2}\right)\theta_0 & |r - r_0| < ct < r + r_0 \\ 0 & \text{otherwise,} \end{cases} \quad (21)$$

$$u_2 = \begin{cases} 0 & t < R/c \\ \frac{1}{4\pi[t^2 - R^2/c^2]^{1/2}} & t > R/c, \end{cases} \quad (22)$$

$$u_3 = \begin{cases} 0 & t < R_1/c \\ \frac{-1}{4\pi[t^2 - R_1^2/c^2]^{1/2}} & t > R_1/c. \end{cases} \quad (23)$$

Now $\phi = \phi^{\text{inc}} + u$, and on comparison of (22) and (7), we see that $\phi^{\text{inc}} = 2u_2$, so that

$$\phi = u_1 + u_2 + u_3. \quad (24)$$

This completes the formal solution of the problem. Since the series converges very slowly, it is not a very practical result as it stands. If the Green's function ϕ were to be used for the determination of the response to some less singular excitation, the series resulting from term-by-term integration of (2) would converge much more rapidly. In special cases, however, the series (21) may be summed, and we shall now consider some examples where this may be accomplished.

There is an apparent difficulty that may be resolved at once. The term u_3 represents a source at the image of the source point whereas there is no such source in the original problem. There is, however, a contribution from u_1 which just cancels the singularity in u_3 at $(r_0, 2\pi - \theta_0)$. To see this, we investigate the behavior of u_1 and u_3 at this point. We have at once that

$$u_3 = -1/4\pi t \quad t > 0,$$

and

$$u_1 = \begin{cases} (c/2\pi r_0) \sum_{n=0}^{\infty} P_n \left[1 - \frac{c^2 t^2}{2r_0^2} \right] \sin^2 \left(n + \frac{1}{2} \right) \theta_0 & 0 < ct < 2r_0, \\ 0 & \text{otherwise.} \end{cases}$$

Using $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, u_1 may be broken up into two series. The series involving $\cos (2n + 1)\theta_0$ is not singular as $t \rightarrow 0$. We therefore consider only

$$u'_1 = \begin{cases} (c/4\pi r_0) \sum_{n=0}^{\infty} P_n \left(1 - \frac{c^2 t^2}{2r_0^2} \right) & 0 < ct < 2r_0 \\ 0 & \text{otherwise.} \end{cases}$$

This series may be summed [6, p. 53]:

$$u'_1 = 1/4\pi t, \quad 0 < ct < 2r_0.$$

Thus, u_1 , as stated above, cancels the singularity of u_3 at the image of the source. Similarly, at (r_0, θ_0) , the u_1 term adds to the term u_2 to give the proper source-like character (i.e. with a coefficient $1/2\pi$ instead of $1/4\pi$) at this point.

The next case to be considered is where $\theta = 3\pi/2$, $\theta_0 = \pi/2$ (see Fig. 3).

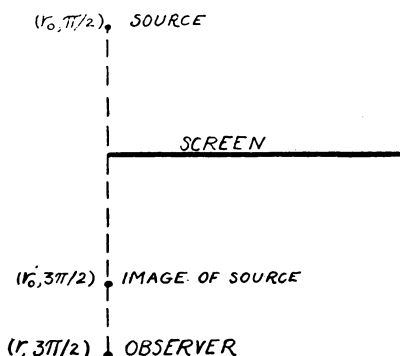


FIG. 3. Configuration for a test of Babinet's principle.

The series may be summed as in the previous example, yielding

$$\phi = \begin{cases} (1/4\pi)\{[t^2 - (r + r_0)^2/c^2]^{-1/2} - [t^2 - (r - r_0)^2/c^2]^{-1/2}\} & ct > r + r_0 \\ 0 & \text{otherwise} \end{cases}$$

Now the problem complementary to the case under consideration has as the boundary condition on the complementary screen, $y = 0$, $x < 0$,

$$\frac{\partial \psi}{\partial n} = 0.$$

The solution here is

$$\psi = \begin{cases} (1/4\pi)\{[t^2 - (r + r_0)^2/c^2]^{-1/2} + [t^2 - (r - r_0)^2/c^2]^{-1/2}\} & ct > r + r_0 \\ 0 & \text{otherwise.} \end{cases}$$

Now Babinet's principle states that the signal and the signal for the complementary problem must add up to the incident excitation. Adding our two solutions, we have

$$\psi + \phi = \begin{cases} (1/2\pi)[t^2 - (r + r_0)^2/c^2]^{-1/2} & ct > r + r_0 \\ 0 & \text{otherwise,} \end{cases}$$

which is indeed the incident field. Thus, Babinet's principle has been verified in the time domain for this special case. Because of the highly discontinuous nature of the solutions, this constitutes a rather severe test of the validity of Babinet's principle in the time domain.

Finally, we shall compute the "charge density"—that is, the discontinuity in $\partial\phi/\partial n$ across the screen, for the case $\theta_0 = \pi$. From (20), the Laplace transform of $\rho(x)$, the charge density, is given by

$$P(x) = (2/\pi x) \sum_{n=0}^{\infty} K_{n+1/2}(pr_0/c) I_{n+1/2}(pr_0/c) (-1)^n \left(n + \frac{1}{2}\right).$$

None of the other terms contribute to $P(x)$ since their normal derivatives are continuous across the screen. This series may be summed [7, p. 366, Eq. 11], giving

$$P(x) = \frac{(1/\pi)(r_0/x)^{1/2} \exp[-(p/c)(x + r_0)]}{r_0 + x}$$

so that

$$\rho(x) = \frac{\delta[t - (1/c)(x + r_0)]}{\pi(r_0 + x)} \cdot (r_0/x)^{1/2}.$$

Thus, the charge on the screen has *no* after-effect, and the "amplitude" of the pulse falls off as $x^{-3/2}$ as we move away from the edge.

IV. Conclusion. We have derived Green's function for the time-dependent wave equation and considered a few of its properties. The method is quite straight forward—two transformations and two inversions—and is readily generalized to the case of the wedge. As we remarked in the introduction, we may also treat the problem of (say) heat flow due to an instantaneous source near a wedge held at constant temperature or near an insulated wedge. The principal change to be made is the substitution of $p^{1/2}$ for p in all of the equations up to Eq. (20). The inversion of the product

$$K_{n+1/2}(p^{1/2}r/c) I_{n+1/2}(p^{1/2}r/c)$$

is not as easy, but can be carried out for any specific n . Although the general term in the series is difficult to determine, the series converges rapidly enough so that this is not important. (Note that the extremely slow convergence of the series (21) is not a fault of the method, but only a consequence of the character of the solution). A paper on this is planned in the near future.

Appendix. Evaluation of V_0 . With a convenient change in notation, we have

$$V_0 = (1/2\pi) \int_0^\infty K_s(x) [K_0(x_0) - K_0\{(x^2 + x_0^2 - 2xx_0 \cos \theta_0)^{1/2}\}] dx/x.$$

We use the addition theorem for Macdonald functions [7, p. 361, Eq. 8], and get

$$2\pi V_0 = \int_0^{x_0} K_s(x) \left[K_0(x_0) - \sum_{m=0}^\infty \epsilon_m K_m(x_0) I_m(x) \cos m\theta_0 \right] \frac{dx}{x} \\ + \int_{x_0}^\infty K_s(x) \left[K_0(x_0) - \sum_{m=0}^\infty \epsilon_m K_m(x) I_m(x_0) \cos m\theta_0 \right] \frac{dx}{x},$$

where $\epsilon_m = 1, m = 0; m = 2, m \geq 1$. Interchanging summation and integration we have:

$$2V_0 = K_0(x_0) \int_0^{x_0} K_s(x) [1 - I_0(x)] \frac{dx}{x} - 2 \sum_{m=1}^\infty K_m(x_0) \cos m\theta_0 \int_0^{x_0} K_s(x) I_m(x) \frac{dx}{x} \\ + \int_{x_0}^\infty K_s(x) [K_0(x_0) - K_0(x) I_0(x_0)] \frac{dx}{x} - 2 \sum_{m=1}^\infty I_m(x_0) \cos m\theta_0 \int_{x_0}^\infty K_s(x) K_m(x) \frac{dx}{x}.$$

The second and fourth integrals are evaluated by means of the indefinite integral for cylinder functions [7, p. 134, Eq. 7]; when this is done, the two series may be combined to give

$$-2x_0 \sum_{m=1}^\infty \frac{\cos m\theta_0}{m^2 - s^2} [K_m K_s I'_m - K_m I_m K'_s - I_m K_s K'_m + I_m K_m K'_s],$$

where the argument x_0 has been suppressed. The second and fourth terms in the square brackets cancel, leaving just $K_s(x_0)$ times the Wronskian of K_m and I_m , which is just $1/x_0$ [7, p. 80, Eq. 19]. Thus, the series in the expression for V_0 combine to give

$$-2K_s(x_0) \sum_{m=1}^\infty \frac{\cos m\theta_0}{m^2 - s^2}.$$

This last series may be summed [11, p. 278, Eq. 13] so that the expression for V_0 now reads

$$2\pi V_0 = K_0(x_0) \int_0^{x_0} K_s(x) [1 - I_0(x)] \frac{dx}{x} - K_0(x_0) \int_{x_0}^\infty K_s(x) [1 - I_0(x)] \frac{dx}{x} \\ + \pi K_s(x_0) \left[\frac{\cos(\pi - \theta_0)s}{s \sin \pi s} - \frac{1}{\pi s^2} \right] + K_0(x_0) \int_{x_0}^\infty K_s(x) \frac{dx}{x} - I_0(x_0) \int_{x_0}^\infty K_s(x) K_0(x) \frac{dx}{x} \\ = K_0(x_0) \int_0^{x_0} K_s(x) [1 - I_0(x)] \frac{dx}{x} + \pi K_s(x_0) \left[\frac{\cos(\pi - \theta_0)s}{s \sin \pi s} - \frac{1}{\pi s^2} \right] \\ + K_0(x_0) \int_{x_0}^\infty K_s(x) I_0(x) \frac{dx}{x} - I_0(x_0) \int_{x_0}^\infty K_s(x) K_0(x) \frac{dx}{x}.$$

The last two integrals may be evaluated as above; this time, we obtain the Wronskian of K_0 and I_0 and another term in $1/s^2$. The Wronskian term cancels the $1/\pi s^2$ term in the above expression, leaving

$$2\pi V_0 = K_0(x_0) \int_0^\infty K_s(x)[1 - I_0(x)] \frac{dx}{x} + \pi K_s(x_0) \frac{\cos(\pi - \theta_0)s}{s \sin \pi s} - (1/s^2)K_0(x_0).$$

To evaluate the remaining integral, consider the contour integral

$$\oint H_s^{(1)}(z)[1 - J_0(z)](dz/z)$$

taken over the contour shown in Fig. 4. In the limit as $\rho \rightarrow 0$ and $R \rightarrow \infty$, the integrals

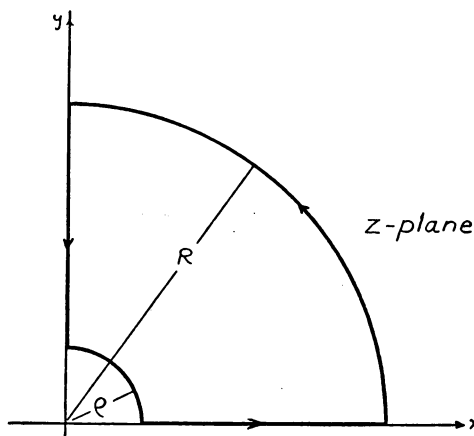


FIG. 4. Contour for evaluation of the integral $\int_0^\infty K_s(y)[1 - I_0(y)] dy/y$.

over the circular arcs contribute nothing (assuming that $|\operatorname{Re}(s)| < 2$; the result for s outside this strip is obtained by analytic continuation). Thus,

$$(2/\pi i)e^{-i\pi s/2} \int_0^\infty K_s(y)[1 - I_0(y)](dy/y) = \int_0^\infty H_s^{(1)}(x)[1 - J_0(x)](dx/x).$$

Equating real parts, we have

$$(-2/\pi) \sin \frac{1}{2} \pi s \int_0^\infty K_s(y)[1 - I_0(y)](dy/y) = \int_0^\infty J_s(x)(dx/x) - \int_0^\infty J_s(x)J_0(x)(dx/x).$$

These integrals have been evaluated elsewhere [7, p. 391, Eq. 1; p. 403, Eq. 2], so that

$$\int_0^\infty K_s(y)[1 - I_0(y)](dy/y) = \frac{-\pi}{2 \sin \frac{1}{2} \pi s} \left[\frac{1}{s} - \frac{2}{\pi s^2} \sin \frac{1}{2} \pi s \right].$$

Putting this into the expression for V_0 , making several cancellations, and reverting to the original notation, we have:

$$V_0 = K_s(pr_0/c) \frac{\cos(\pi - \theta_0)s}{2s \sin \pi s} - K_0(pr_0/c) \frac{1}{4s \sin \frac{1}{2} \pi s}.$$

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