THE APPLICATION OF LIMIT ANALYSIS TO THE DETERMINATION OF THE STRENGTH OF BUTT JOINTS*

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Summary. The technique of limit analysis is applied to determine upper and lower bounds for the tensile strength of a butt joint consisting of a thin layer of adhesive joining the parallel flats of two rigid adherends. The adhesive is assumed to be an elastic-perfectly plastic material which yields when the maximum shear stress reaches a critical value. The methods used apply to any joint with a convex area of cross-section. Particular application is made to joints whose cross-sections are circular, rectangular, or a polygon circumscribed about a circle.

1. Introduction. It is known experimentally (see [1]**, for example) that the strength of a butt joint of a hardened adhesive between two rigid adherends is inversely proportional to the thickness of the adhesive layer when the layer is thin. Various theories have been suggested to explain this phenomenon and critical reviews of these theories can be found elsewhere [2, 3]. The theory of the present paper rests on the assumption that the adhesive can be represented with sufficient accuracy by an elastic-plastic material which yields under constant stress when the maximum shear stress reaches a certain critical value. Under this simplifying assumption, the problem of the determination of the stress distribution in a butt joint under tension is still one of considerable difficulty. However the assumption enables the technique of limit analysis [4, 5, 6] to be used to predict the strength of the joint. In Secs. 3 and 4 below, particular application is made to joints whose sections are rectangular, circular, or a polygon circumscribed about a circle, but the analysis can be modified so that it applies to any joint with a convex area of cross-section. An approximate analysis for a circular joint has been made previously by Kachanov [7].

The related problem of the plastic compression of a layer between two rough plates has also been treated by approximate analysis [8, 9]. Since the yield condition used here for the adhesive is independent of hydrostatic pressure, reversal of the stresses in the analysis below gives results which apply to the related problem, and these results agree with the corresponding results in [8, 9].

2. Limit analysis and statement of the problem. For an assemblage of rigid and of elastic-perfectly plastic bodies, the term collapse will be used here to describe conditions under which plastic flow would occur under constant loads if the accompanying changes in the geometry of the assemblage were neglected. Assuming that the boundary conditions are of the stress type (i.e. each component, T_x , T_y or T_z , of the surface traction is given except where the corresponding velocity component, v_x , v_y or v_z , or the corresponding relative velocity component at the interface of an assemblage is prescribed to be zero), the limit analysis theorems [4, 5, 6] provide a method for determin-

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ing whether or not collapse will occur at any stage in a given loading program. The limit analysis theorems of Ref. 6 which will be used below are outlined briefly here.

A stress field is said to be statically admissible if the stress field is in equilibrium and satisfies the stress boundary conditions. The stress fields employed in the following consist of regions of constant stress separated by surfaces across which the stresses are discontinuous. These fields will be in equilibrium if the expression

$$\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z , \tag{1}$$

and each of the two similar expressions, is continuous across a surface of stress discontinuity, n_x , n_y and n_z being the components of the unit normal to the surface. A stress field is said to be safe if the yield condition is nowhere violated.

From any velocity field v_x , v_y , v_z , strain rates can be derived in the usual way;

$$\epsilon_x = \frac{\partial v_x}{\partial x} , \cdots , \qquad \gamma_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} , \cdots . \tag{2}$$

If these strain rates are treated as purely plastic strain rates, the internal rate of dissipation of energy can be calculated, provided that the strain rates satisfy any condition imposed by the yield condition. Here the yield condition is assumed to be independent of the hydrostatic component of the stress tensor and imposes the incompressibility condition,

$$\epsilon_x + \epsilon_y + \epsilon_z = 0, \tag{3}$$

on plastic strain rates. A velocity field is called kinematically admissible if it satisfies the incompressibility condition and satisfies the velocity boundary conditions. Such a velocity field is said to be a kinematically admissible state of collapse if the total internal rate of dissipation of energy is not greater than the rate at which the applied tractions do work on the velocities of their points of application.

The following theorems have been formulated for the case of all surface tractions increasing in proportion:

Theorem 1. Collapse will not occur until the largest values of the surface tractions are reached for which it is possible to find a safe statically admissible stress field.

Theorem 2. Collapse will occur under the smallest values of the surface tractions for which it is possible to find a kinematically admissible state of collapse.

Theorems 1 and 2 can be used to determine lower and upper bounds, respectively, for the collapse values of the surface tractions.

For a material obeying Tresca's yield criterion, plastic flow can occur under constant maximum shearing stress k, and safe statically admissible stress fields in the material must nowhere involve a shearing stress greater than k. Also, for this material, the internal rate of dissipation of energy per unit volume due to a plastic strain rate is $2k \max | \epsilon |$, where $\max | \epsilon |$ is the numerically largest principal component of the plastic strain rate [10]. Thus kinematically admissible collapse states must be such that

$$\int_S (T_x v_x + T_y v_y + T_z v_z)\, dS \geq \int_V 2k \max | \epsilon | \, dV, \tag{4}$$

where S is the bounding surface and V the volume of the body or assemblage of bodies, and where $\max | \epsilon |$ refers to the strain rates derived from the velocity field. Under the Tresca criterion, discontinuities in the tangential velocity across fixed surfaces are

permissible in kinematically admissible velocity fields. At a surface of discontinuity, energy is dissipated at the rate $k\Delta v$ per unit surface area, where Δv is the magnitude of the relative change in velocity across the surface, and this must be taken into account in evaluating the total internal rate of dissipation of energy.

For the purposes of this report, a butt joint is treated as a thin layer of adhesive joining the parallel flats of two rigid adherends. The free ends of the adherends are loaded by two equal and opposite tensile forces of magnitude T, having a common line of action which is normal to the joint and which passes through the centroid of the cross-sectional area of the joint. The adhesive is assumed to be an elastic-perfectly plastic material obeying Tresca's yield condition. The tensile force T across the joint is increased until the collapse load is reached and the joint fails due to plastic yielding of the adhesive layer. The adherends and the adhesive layer constitute an assemblage of bodies to which the limit analysis theorems can be applied to determine the collapse value of the load T. Statically admissible stress fields in the layer must be such that there is no shearing stress on the mid-plane of the layer because of symmetry, and also the resultant tensile force on the mid-plane must act through the centroid of the area of the mid-plane. At the interfaces of the adhesive and the adherends, the tractions can have any value compatible with safe statically admissible stress fields in the adhesive, since the adherends are rigid and since there is complete attachment between the adhesive and the adherends. Kinematically admissible velocity fields must be such that there is no relative motion between the adherends and adhesive immediately adjacent to the adherends, which move as rigid bodies. However, surfaces of tangential velocity discontinuity between the adhesive and the adhesive immediately adjacent to the adherends are permissible.

3. Lower bounds. In this section, a method is given for obtaining lower bounds for the collapse value of the tensile load T for butt joints which have a convex area of cross-section. The stress fields employed are derived from a stress field used previously to obtain a lower bound for the average indentation pressure in the plastic indentation of a layer by a circular punch [11].

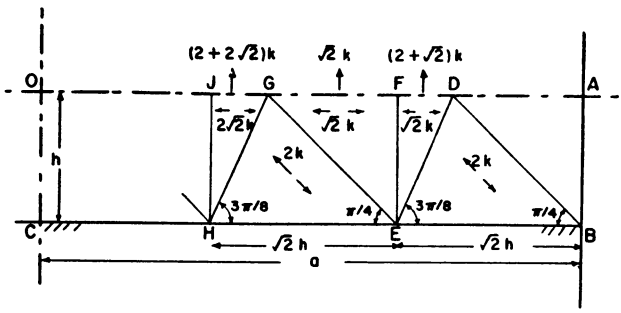

FIG. 1. Stress field for plane strain problem.

Figure 1 shows the plane strain version of the stress fields used. In the figure, OA is the mid-plane of the adhesive layer of thickness $2h$ and BC is the upper surface of the lower adherend of width $2a$. The field is symmetrical about the mid-plane $0A$ of the layer and about the central plane OC. Region ABD is unstressed and lines BD, DE, EF, etc. are lines of stress discontinuity. The tensile stress of amount $2k$ in region BDE parallel

to BD produces a vertical tension $(2 + 2^{1/2})k$ and a horizontal tension $2^{1/2}k$ in region DEF, where k is the yield shear stress of the adhesive. The stress in region EFG is taken to be a hydrostatic tension of amount $2^{1/2}k$ in order that a tension $2k$ parallel to EG can be superimposed in region EGH. This stress increases the stress across GJ to $(2 + 2^{3/2})k$. By repeated application of this process the tension across the mid-plane is increased towards the center O.

The extension of the field depicted in Fig. 1 to a butt joint of circular cross-section is shown in plan in Fig. 2. From strip elements of area on the conical surface through DE in Fig. 1, "legs" of material originate and carry tensions of amount $2k$ inclined at an angle of amount $\pi/4$ to the horizontal. Thus $DEE'D'$ in Fig. 2 is a strip element of

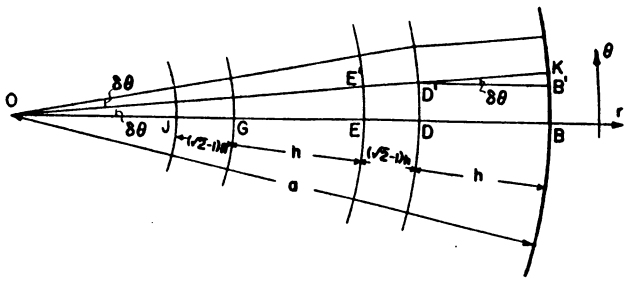

Fig. 2. Plan of stress field for circular joint.

area on the conical surface through DE in Fig. 1 subtending an angle $\delta\theta$ at the axis OC. This strip element generates the "leg" $DEBB'E'D'$ which carries a tension $2k$ in the direction BD. The material vertically above the conical surface through DE is stressed by a vertical tension $(2 + 2^{1/2})k$ and equal radial and circumferential tensions $2^{1/2}k$. The material lying between the cylindrical surface through EF and the conical surface through EG is stressed by a hydrostatic tension $2^{1/2}k$. The process of increasing the tension across the mid-plane towards the center of the punch is repeated, so that the annulus in the mid-plane bounded by the circles through G and J carries a normal stress $(2 + 2^{3/2})k$ and so on. Further, triangular elements of area in the mid-plane, such as $B'D'K$ in the plan, do not lie vertically above "legs" of material carrying a tension $2k$. The tension over these areas can be increased by adding a vertical tension $2k$ in the triangular prisms of material below the areas.

The field described nowhere violates the yield condition and is a statically admissible stress field, since it satisfies the symmetry conditions on the mid-plane. It therefore provides a lower bound for the collapse load of the joint. When $a/h = n2^{1/2}$, where n is an integer, the process of increasing the tension towards the center is carried out n times, and the average tension t_L over the mid-plane is found to be given by

$$\frac{t_L}{k} = 1 - 2^{-1/2} + \frac{1}{3}\frac{a}{h} + \left(2^{1/2} - \frac{2}{3}\right)\frac{h}{a} \cdot \tag{5}$$

The product of t_L and the area πa^2 of the joint is a lower bound for the collapse load T.

The stress field outlined above can be modified so that it applies to any joint which has a convex area of cross-section. When the glued section is a rectangle with sides $2a$, $2b$, where $b \geq a$, the average tension t_L over the mid-plane is found to be given by

$$\frac{t_L}{k} = 1 - 2^{-1/2} + \frac{1}{3}\frac{a}{h}\left(\frac{3}{2} - \frac{1}{2}\frac{a}{b}\right) + \left(2^{1/2} - \frac{2}{3}\right)\frac{h}{b}, \tag{6}$$

for $a/h = n2^{1/2}$, where as before $2h$ is the thickness of the adhesive layer and n is an integer. The value $4abt_L$ is a lower bound for the collapse load T.

For a square cross-section, $a = b$ and the value (6) reduces to the value (5). Indeed it can be shown that for any joint whose cross-section is a polygon circumscribed about a circle of radius a, the average tension t_L over the joint provided by the stress field is given by (5) for $a/h = n2^{1/2}$.

An alternative stress field for a rectangular joint can be derived from a stress field used previously in [11] to obtain a lower bound for the average indentation pressure in the plastic indentation of a layer by a square punch. The field provides a value t_L for the average tension over the joint given by

$$\frac{t_L}{k} = 1 + \frac{1}{3}\frac{a}{h}\left(\frac{3}{2} - \frac{1}{2}\frac{a}{b}\right) + \frac{h}{a}\left(\frac{5}{6}\frac{a}{b} - \frac{1}{2}\right) + \frac{1}{3}\frac{h^2}{ab}, \tag{7}$$

for $a/h = 1 + 2^{1/2} + n2^{3/2}$, $2a$ and $2b$ ($b \geq a$) being the sides of the rectangle. The value (7) for t_L is greater than the value (6) but the percentage difference between the values is small for large values of a/h.

4. Upper bounds. In applying Theorem 2 of Sec. 2 to obtain upper bounds for the collapse value of the tensile load T across a butt joint, the procedure is as follows. A mode of deformation of the adhesive layer is assumed which is compatible with the incompressibility condition (3) and the condition that the adherends move as rigid bodies. The total rate of plastic dissipation of energy due to this velocity field is then calculated. With this kinematically admissible velocity field the rate at which work is done by tensile loads T applied to the adherends can be found, and the value, T_U say, of T is chosen so that the rate at which the loads T_U do work is equal to the rate of plastic dissipation of energy in the layer. Under the loads T_U the velocity field is a kinematically admissible state of collapse and it follows from Theorem 2 that T_U is an upper bound for the collapse value of the load T.

In the velocity fields employed below, it is assumed that the adherends have equal and opposite velocities V normal to the joint so that the adherends separate at a velocity $2V$. The z-axis is taken normal to the joint area and the mid-plane of the adhesive layer is taken to be the plane $z = 0$. The velocity fields are symmetrical about the plane $z = 0$ so that only the motion in the upper half of the layer, the region $0 \leq z \leq h$, need be considered. The velocity component v_z is zero on $z = 0$ because of symmetry and has the value V on the flat $z = h$ of the upper adherend, since v_z must be continuous across $z = h$. These conditions on v_z are satisfied by assuming that

$$v_z = Vz/h, \qquad 0 \leq z \leq h. \tag{8}$$

The velocity components v_x and v_y in the layer must then satisfy the incompressibility equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -\frac{V}{h}, \tag{9}$$

and for simplicity v_x and v_y are assumed to be independent of z. With cylindrical polar co-ordinates (r, θ, z), Eq. (9) becomes

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = -\frac{V}{h}, \tag{10}$$

where v_r and v_θ are the radial and circumferential velocity components, assumed to be functions of r and θ only.

For the joint of circular cross-section we take

$$v_r = -Vr/2h, \qquad v_\theta = 0, \qquad 0 \leq z < h, \tag{11}$$

where the origin of the coordinate system is at the center of the middle section. On $z = h$, v_r and v_θ are zero. Energy is dissipated in the layer due to the non-zero strain rates and also in the velocity discontinuity surface between the adhesive and the adhesive immediately adjacent to the flats $z = \pm h$ of the adherends. A straightforward calculation shows that the velocity field will be a kinematically admissible collapse state if the average tension over the mid-plane is greater than t_U, where

$$\frac{t_U}{k} = 2 + \frac{1}{3}\frac{a}{h}, \tag{12}$$

a being the radius of the cross-section. By Theorem 2 the product of t_U and the area πa^2 of the cross-section is an upper bound for the collapse value of the load T. The percentage difference between the upper bound (12) and the lower bound (5) for the average tension over the joint tends to zero as a/h tends to infinity.

The simple velocity field (11) can also be used for non-circular cross-sections. For a joint with a square cross-section of side $2a$, the field is a kinematically admissible collapse state if the average tension over the joint is not less than t_U given by

$$\frac{t_U}{k} = 2 + 0.383\frac{a}{h}\cdot \tag{13}$$

For very large values of a/h, the difference between the upper bound (13) and the lower bound (5) is 15 per cent of the lower bound.

In order to improve the upper bound (13) for a square section, and also for application to other sections, more sophisticated velocity fields are required. With a view to constructing velocity fields for polygonal sections, consider a right prism of adhesive whose section is a right-angled triangle, triangle OAB in Fig. 3, and which is bounded by the planes $z = 0$, $z = h$. The coordinate system is taken as shown in the figure and v_z is again given by Eq. (8). The components v_r and v_θ are zero on $z = h$. In the region $0 \leq z < h$, the incompressibility condition (10) is satisfied by the expressions

$$v_r = -\frac{Vr}{2h}\frac{df}{d\theta}, \qquad v_\theta = -\frac{Vr}{h}(\theta - f), \tag{14}$$

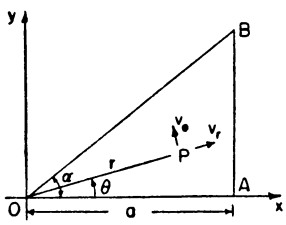

FIG. 3. Velocity field in triangular region.

where f is a function of θ only. We shall assume that v_θ is zero on the sides $\theta = 0$ and $\theta = \alpha$ of the prism since this will facilitate the fitting together of fields for a polygonal cross-section. The total rate D of dissipation of energy in the prism and in the discontinuity surface $z = h$ can, in principle, be minimised with respect to the function $f(\theta)$, subject to $f(0) = 0$, $f(\alpha) = \alpha$. This will not be done here, however. Taking $f(\theta)$ to be given by a linear function of θ gives the field (11). Assuming $f(\theta)$ to be a quadratic function of θ gives

$$f(\theta) = B\theta\left(1 - \frac{\theta}{\alpha}\right) + \frac{\theta^2}{\alpha} \tag{15}$$

and the associated rate D of dissipation of energy can be minimised with respect to the arbitrary constant B. The minimum value of D considered as a function of B is near the point $B = 2$ and taking $B = 2$ provides results sufficient for the purposes of this paper. Substituting (15) in (14) and setting $B = 2$ gives the components

$$v_r = -\frac{Vr}{h}\left(1 - \frac{\theta}{\alpha}\right), \qquad v_\theta = \frac{Vr}{h}\,\theta\left(1 - \frac{\theta}{\alpha}\right). \tag{16}$$

It should be noted that the value of v_r on $\theta = 0$ and $\theta = \alpha$ is independent of α. It is found that the total rate D of dissipation of energy is given by

$$\frac{D}{kA} = 1 + M(\alpha) + \frac{1}{3}\frac{a}{h}K(\alpha), \tag{17}$$

where

$$K(\alpha) = \frac{2}{\tan\alpha}\int_0^\alpha \sec^3\theta\left(1 - \frac{\theta}{\alpha}\right)(1 + \theta^2)^{1/2}\,d\theta, \tag{18}$$

$$M(\alpha) = \frac{1}{\tan\alpha}\int_0^\alpha \sec^2\theta\, g(\theta)\,d\theta, \tag{19}$$

and where $A = \frac{1}{2}a^2 \tan\alpha$ is the cross-sectional area of the prism. In expression (19), $g(\theta)$ is given by

$$g(\theta) = \max\left\{1, \left[\frac{1}{\alpha^2} + \left(1 - \frac{2\theta}{\alpha}\right)^2\right]^{1/2}\right\}.$$

The values of $K(\alpha)$ and $M(\alpha)$ for values of α between 0° and 90° are shown in the table to three decimal places. As α tends to zero through positive values, $M(\alpha)\tan\alpha$ tends to unity.

α:	0°	10°	20°	30°	40°	45°
$K(\alpha)$:	1.000	1.000	1.000	1.000	1.000	1.001
$M(\alpha)$:	∞	5.759	2.922	1.996	1.547	1.402
α:	50°	60°	70°	80°	90°	
$K(\alpha)$:	1.001	1.004	1.011	1.037	1.185	
$M(\alpha)$:	1.290	1.141	1.089	1.084	1.185	

For a square joint of side $2a$, the upper half of the adhesive layer consists of eight

triangular prisms in which $\alpha = 45°$. With the field (16) in each prism the upper bound t_U obtained for the average tension across the joint is given by

$$\begin{aligned} \frac{t_U}{k} &= 1 + M\left(\frac{\pi}{4}\right) + \frac{1}{3}\frac{a}{h}K\left(\frac{\pi}{4}\right) \\ &= 2.402 + \frac{1.001}{3}\frac{a}{h}\cdot \end{aligned} \tag{20}$$

This bound is 0.1 per cent above the lower bound (5) for large values of a/h.

A polygon of n sides circumscribed about a circle of radius a consists of $2n$ triangles such as triangle OAB in Fig. 3, subtending angles $\alpha_1, \alpha_2, \cdots, \alpha_n$ at the center O of the circle, where $\sum_1^n \alpha_r = \pi$. The use of the field (16) in each triangular prism of the adhesive layer for a joint with such a section then provides the upper bound

$$\frac{t_U}{k} = 1 + \frac{\sum_1^n M(\alpha_r)\tan\alpha_r}{\sum_1^n \tan\alpha_r} + \frac{1}{3}\frac{a}{h}\frac{\sum_1^n K(\alpha_r)\tan\alpha_r}{\sum_1^n \tan\alpha_r} \tag{21}$$

for the average tension across the joint. The coefficient of the term $1/3\ a/h$ in (21), which is the important term for thin layers of adhesive, is less than $K(\alpha_{max})$, where α_{max} is the numerically greatest of the angles $\alpha_1, \alpha_2, \cdots, \alpha_n$. Thus the difference between the upper bound (21) and the lower bound (5) is always less than 19 per cent of the lower bound for large values of a/h. The term independent of a/h in (21) may become appreciable if a number of the angles α_r are small and if a/h is not too large. In this case a better upper bound may be obtained by using the field (11) in the prisms subtending small angles at O.

For elongated sections it may be advantageous to assume that regions of the adhesive move in a plane strain manner. As an illustration consider a joint whose section is a

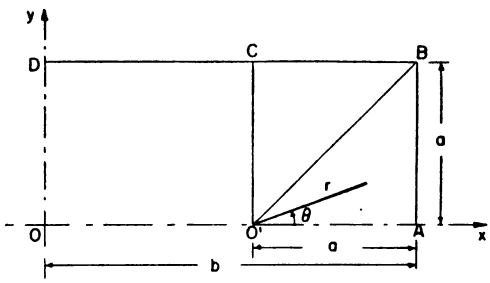

FIG. 4. Plan for rectangular joint velocity field.

rectangle of sides $2a$, $2b$, where $b \geq a$. Figure 4 shows one quarter of the rectangle. The velocity field is symmetrical about the lines OA, OD parallel to the sides and passing through the center O of the rectangle. In region $O'AB$ the components v_r, v_θ have the values (16), where the origin of the polar coordinates r,θ is taken at O'. The field in region $O'BC$ is the image of the field in region $O'AB$ with respect to the line $O'B$. In region $OO'CD$ the plane strain motion

$$v_x = 0, \qquad v_y = -Vy/h$$

is assumed. The field provides the upper bound

$$\frac{t_U}{k} = 2 + 0.402\,\frac{a}{b} + \frac{1}{3}\frac{a}{h}\left[\frac{3}{2} - 0.499\,\frac{a}{b}\right] \tag{22}$$

which is close to the lower bound (7).

The upper and lower bounds obtained above determine the strength of butt joints whose sections are circular, rectangular, or a polygon circumscribed about a circle with sufficient accuracy for practical purposes. The strength of the joint is inversely proportional to the thickness of the adhesive layer for thin layers. The bounds indicate that, for a given cross-sectional area and a given thickness of adhesive, the circular joint is the strongest of all joints with a convex area of cross-section.

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