ENERGY DISSIPATION AND LINEAR STABILITY*

BY

AUREL WINTNER

The Johns Hopkins University

1. Let A be an n by n matrix, A^* its Hermitian adjoint, and A_* the matrix $(A + A^*)/2$. In the real field, this means that $A = A_* + S$, where A_* is (real and) symmetric and S is skew-symmetric. Hence, if $C(x, y)$ denotes the (real) bilinear form belonging to $C(=A, A_*, \cdots)$, then, since $S(x, x)$ vanishes identically, $A(x, x) = A_*(x, x)$ holds for every vector x with real components. The restriction to the real field is made only in order to simplify the notations (and, incidentally, it does not involve any restriction at all, since n can be replaced by $2n$).

If A is a continuous function $A(t)$, where $t_0 \leqq t < \infty$, and if $x(t)$ is any solution vector of

$$x' = A(t)x \tag{1}$$

($x' = dx/dt$) which is distinct from the trivial solution $x(t) \equiv 0$, then, as is well-known, $x(t) \neq 0$ holds for every t. The components of $x(t)$ can be assumed to be real, since $A(t)$ is supposed to be real, hence the real and imaginary parts of a complex solution $x(t)$ of (1) are solution vectors. Let (1) be called stable if $|x(t)| <$ const., as $t \to \infty$, holds for the length of every solution of (1) (and for some constant, which depends on the integration constants determining the solution). On the other hand, let

$$x'' = F(t)x \tag{2}$$

($x'' = d^2x/dt^2$) be called stable if not only $|x(t)| <$ const. but also[1] $|x'(t)| <$ Const. holds for all solutions of (2).

2. For a fixed t, let $\lambda = \lambda(t)$ denote the least, and $\mu = \mu(t)$ the greatest, of the n eigenvalues of the symmetric part[2] $A_* = A_*(t)$ of the coefficient matrix of (1). Then the (unilateral) restriction

$$-\infty \leqq \limsup_{T\to\infty} \int_{t_0}^{T} \mu(t)\, dt < \infty \tag{3}$$

is sufficient in order that (1) be stable. With

$$\lambda = \min_{|y|=1} A(y, y), \qquad \mu = \max_{|y|=1} A(y, y) \tag{4}$$

(where t is fixed in $A = A(t)$ when the min and the max are formed). I found this criterion quite a while ago[3] as a by-product of considerations on non-linear generalizations of (1).

*Received Aug. 17, 1956.

[1]Concerning various types of linear stability in the scalar case, cf. A. Wintner, Quart. Appl. Math. **13** (1955) 192-195.

[2]The point is that only this, the symmetric, part of the coefficient matrix of (1) enters into the stability criterion (3). For a manifestation of the same situation in a different result, cf. A. Wintner, Amer. Journ. Math. **71** (1949) 362-366.

[3]A. Wintner, Amer. Journ. Math. **68** (1946) 557-559. The result and the method of proof were rediscovered by T. Ważewski, Studia Math. **10** (1948) 48-59, and by others. Cf. R. Conti, Rivista di Mat. Univ. Parma **6** (1955) 3-35, where also further references will be found.

Since the proof is very short, and since certain variants of it will be needed when dealing with (2) below, let it be repeated here: On the one hand, (4) is clear from the identity $A(y, y) = A_*(y, y)$ (which is equivalent to the definition of A_*). On the other hand, scalar multiplication of (1) by $x(t)$ shows that $(|x(t)|^2)'/2$ is identical with $A(x, x)$, where $A = A(t)$, $x = x(t)$. But if the trivial solution $x(t) \neq 0$ is excluded, then division by $|x(t)|^2$ is allowed, and so (4) shows that

$$\lambda(t) \leqq (\log |x(t)|)' \leqq \mu(t). \tag{5}$$

Finally, if (3) is assumed, then it is clear from (5) that $\log |x(t)|$, hence $|x(t)|$, must stay bounded as $t \to \infty$.

The simplest instance of (3) is $\mu(t) \leqq 0$, that is, the assumption that the symmetric part $A_*(t)$ of $A(t)$ is non-positive definite at every t. But more than the stability of (1) can then be asserted.[4] In fact, every $|x(t)|$ must then have a finite limit ($\geqq 0$) as $t \to \infty$, since (5) shows that $\log |x(t)|$ is a non-increasing function if $\mu(t) \leqq 0$ throughout, and so the assertion follows from the existence of the lower bound 0 for the length $|x(t)|$. This deduction is mentioned here because a more elaborate form of it will be needed below.

3. The following considerations will replace (1) by (2). For the sake of concreteness, it will be assumed that the matrix $F = F(t)$ is symmetric (that is, that $F = F_*$ for every t), so that (2) is a (non-conservative, but reversible) linear Lagrangian system, with n degrees of freedom, having the kinetic and potential energies $|x'|^2/2$ and $-F(x, x)$ respectively, and the latter contains t explicitly (there is, of course, no loss of generality in the normalization $|x'|^2/2$ which, if all masses are positive and constant, puts all of them equal to 1). The assumptions will be placed on the (time-dependent) eigenvalues of $F = F_*$. Actually, it will be clear from the proofs that the symmetry restriction $F = F_*$ can be omitted if the assumptions are placed on the symmetric part, $F_*(t)$, of the coefficient matrix of (2).

The energy along a path $x = x(t)$ (which actually must be thought of in phase space, rather than in configuration space) is the scalar $h = h(t)$ defined by

$$2h = |x'|^2 - F(x, x). \tag{6}$$

But $h(t) =$ const. does not hold (unless $F(t) =$ Const.). There is[5] however a simple explicit rule for the dissipation of the energy which, when subject to appropriate assumptions, can be put to about the same use as the conservation of the energy or its "virial" formulation in the classical results (Dirichlet, Jacobi, Hill, Liapounoff) dealing with stability in the conservative case. In the present case, the explicit form of the law of dissipation is particularly simple[6]; it states that, along any solution path $x = x(t)$ of (2), the energy $h = h(t)$ must vary so as to render the relation

$$dh = D(x, x), \quad \text{where} \quad D = dF \quad \text{and} \quad D = D(t), \tag{7}$$

an identity in t.

---

[4]In this regard, cf. A. Wintner, Quart. Appl. Math. **8** (1950) 102-104.

[5]For the general (not necessarily linear) case, cf. A. Wintner, Phil. Mag. **39** (1948) 722-728.

[6]Cf. the reference in the preceding footnote.

If $F(t)$ is just continuous, then (7), as it stands, is meaningless, but it becomes meaningful if it is meant in terms of Stieltjes integrations, with reference to the "measures" $dh(t)$ or $D(t)$. For, on the one hand, a Riemann-Stieltjes integral $\int f(t)dg(t)$ exists whenever $g(t)$ is continuous and $f(t)$ is continuously differentiable (hence, of bounded variation) and, on the other hand, not only $f(t) = x(t)$ but also $f(t) = x'(t)$ is continuously differentiable if $x(t)$ is any solution of (2), if $F(t)$ is just continuous.

If the coefficient matrix F of (2), instead of being just continuous, has a continuous derivative (for the sake of shortness, this will be assumed from now on), then (7) can be written in the form

$$h' = F'(x, x), \quad \text{where} \quad h = h(t), \qquad F' = F'(t) \tag{7 bis}$$

(the existence of a continuous derivative h' is part of the statement). In fact, scalar multiplication of (2) by x' leads from (6) to (7 bis), since $x' \cdot x'' = (| x' |^2)'/2$.

4. The following fact is a corollary: If the quadratic form belonging to the (symmetric) matrix $F'(t)$ is non-positive [non-negative] definite at a fixed t, then the energy $h(t)$ of every solution vector $x = x(t)$ of (2) is a non-increasing [non-descreasing] function at that t.

In fact, if $\alpha = \alpha(t)$ denotes the least, and $\beta = \beta(t)$ the greatest, eigenvalue of $F'(t)$ at a fixed t, then (7 bis) shows that

$$\alpha(t) \mid x(t) \mid^2 \leqq h'(t) \leqq \beta(t) \mid x(t) \mid^2. \tag{8}$$

But (8) contains the assertion, since the assumption is that $\beta(t) \leqq 0$ [or $\alpha(t) \geqq 0$].

Corresponding to the notation in Section 1, let $\lambda = \lambda(t)$ be the least, and $\mu = \mu(t)$ the greatest, eigenvalue of $F = F(t)$ itself; so that (even if x is not a solution vector)

$$\lambda(t) \mid x \mid^2 \leqq F(x, x) \leqq \mu(t) \mid x \mid^2, \quad \text{where} \quad F = F(t). \tag{9}$$

Since $\mu(t) < 0$ [or $\lambda(t) > 0$] means that $F(t)$ is negative [positive] definite, an adaptation of the proof of criterion (3) for the stability of (1) will now lead to the following criterion (i) for the stability of (2):

(i) *If the coefficient matrix $F(t)$ of* (2) *is strictly negative definite for large t, and if $F'(t)$ is non-positive definite at every fixed large t, then* (2) *is stable.*

It is understood that by the first assumption of (i) is meant the existence of a positive constant ϵ satisfying

$$\max_{|y|=1} F(y, y) = \mu = \mu(t) \leqq -\epsilon < 0. \tag{10}$$

On the other hand, the second assumption of (i) means that $\beta(t) \leqq 0$ holds throughout and so, since (8) implies that $h'(t) \leqq 0$, the energy $h(t)$ tends either to a finite limit $h(\infty)$ or to $-\infty$. But (10) prevents the second contingency and, what is more, ensures that $h(\infty) \geqq 0$. In fact, since (6) and (9) imply that $2h \geqq | x' |^2 - \mu | x^2 |$, it follows from (10) that

$$2\epsilon^{-1}h(t) \geqq \mid x(t) \mid^2 + \mid x'(t) \mid^2, \tag{11}$$

if, without loss of generality, ϵ is chosen to be less than 1. But (11) shows that $h(t)$ cannot become negative. Hence there exists a finite $h(\infty) \geqq 0$. It follows therefore from (11) that $x(t)$ and $x'(t)$ stay bounded as $t \to \infty$. This proves (i).

5. Suppose, for a moment, that (2) is of a single degree of freedom. Then (2) can be written in the form

$$x'' + \omega^2(t)x = 0 \tag{12}$$

if $-\omega^2 = F \leqq 0$ for all t, as in (i). Suppose that

$$\omega(t) \to \infty \quad \text{as} \quad t \to \infty. \tag{13}$$

Then one would expect that the successive waves of every solution $x(t)$ of (12) (waves the widths of which must tend, as $t \to \infty$, to 0 by virtue of Sturm's comparison theorem) will compel the graph of $x(t)$ to be such as to satisfy

$$x(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{14}$$

But it is known that this turns out to be wrong, since even[7] $\limsup |x(t)| = \infty$ is compatible with (13), and that (14) follows from (13) only under supplementary conditions on the regularity of the growth of $\omega(t)$, the simplest such condition being the monotony of $\omega(t)$,

$$d\omega(t) \geqq 0. \tag{15}$$

The known proofs[8] of the sufficiency of (13) and (15) for (14) depend very much on Sturm's separation theorem and cannot, therefore, be adjusted to an extension to the non-scalar case of (2), the case of more than a single degree of freedom. It will now be shown that the more primitive approach, used above, can readily be modified so as to fill in this gap, as follows:

(ii) *If the greatest of the n eigenvalues of $F(t)$ tends to $-\infty$ as $t \to \infty$, and if $F'(t)$ is non-positive definite at every fixed t, then* (14) *holds for every solution vector of* (2).

Needless to say, the assumptions of (ii) reduce to (13) and (15) in the scalar case ($n = 1$). It is also clear that the assumptions of (ii) imply those of (i). But it was seen in the proof (i) that there exists a finite limit $h(\infty) \geqq 0$.

The first assumption of (ii) is that, if ϵ is an arbitrarily small positive number, and if t is large enough, say $t > t_\epsilon$, then $-\mu(t) < 1/\epsilon$. It follows therefore from (9) and (6) that

$$2h(t) \geqq |x'(t)|^2 + |x(t)|^2/\epsilon \quad \text{if} \quad t_\epsilon < t < \infty.$$

Since $|x'(t)|^2 \geqq 0$, and since there exists a finite $h(\infty) \geqq 0$, this implies that

$$2\epsilon h(\infty) \geqq \limsup_{t\to\infty} |x(t)|^2. \tag{16}$$

But the constant $h(\infty)$ and the expression on the right of (16) are independent of ϵ' and ϵ can be chosen arbitrarily small. Hence the assertion, (14), of (ii) follows from (16)

6. The following variant of (i) will now be proved:

If $F(t)$ is strictly negative definite, and if the n eigenvalues of $F'(t)$ (or, what is the same thing, if both functions

$$\min_{|y|=1} F'(y, y), \qquad \max_{|y|=1} F'(y, y) \tag{17}$$

of t) are absolutely integrable,[8] then (2) is stable.

Note that (iii), in contrast to the unilateral assumption of (i), does not assume that the (internal) dissipating action is *steadily* decreasing. In fact, the absolute integrability

[7]For an explicit example, cf. A. Wintner, Journ. Appl. Phys. **18** (1947) 941-942, where a simple proof for (14) under the supplementary assumption (15) is also given.

The restriction (15) is a particular case of a more inclusive condition, formulated by G. Armellini (1935) and proved by L. Tonelli and G. Sansone (1936), a condition (on the "regularity of the growth" of the coefficient function) which, when added to (13), is sufficient to ensure (14) for all solutions of (12); cf. the detailed presentation in Sansone's *Equazioni differenziali nel campo reale*, 2nd ed. (1949), vol. 2, pp. 52-67, where (on p. 53 and p. 61) further references will be found.

[8]Cf. the preceding footnote.

of both functions (17) of t is compatible with both of them coming arbitrarily close to $-\infty$ *and* ∞ as $t \to \infty$.

Actually, the last criterion can greatly be improved, since the assumption placed on the first of the two functions (17) can be omitted entirely, and the *negative part* of the second of the functions (17) does not matter either (this agrees with (i), where that negative part is arbitrary but the positive part is missing). In other words, all that is needed is

$$\int_{t_0}^{\infty} \gamma(t)\, dt < \infty, \tag{18}$$

where, if $\beta(t)$ has the same meaning as in (8),

$$\gamma = \max(0, \beta) \tag{19}$$

(for a fixed t); so that the situation is as follows:

(i bis) *If $F(t)$ is strictly negative definite, and if $\beta(t)$, the greatest eigenvalue of $F'(t)$, satisfies the unilateral restriction* (18), *where $\gamma(t)$ is defined by* (19), *then* (2) *is stable.*

The proof of this extension, (i bis), of (i) requires a combination of the proof of (i) with an adaptation of a device used in another context[9]. First, it is seen from (19) and (8) that, if a t-value is called of the first or of the second kind according as $h'(t) \leqq 0$ or $h'(t) > 0$, then $|h'(t)| \leqq g(t)|x(t)|^2$ for t-values of the second kind. But since $F(t)$ is assumed to be strictly negative definite, it follows, as in the proof of (i), that (11) holds for every t and for a fixed $\epsilon > 0$. Hence, for t-values of the second kind, $|h'(t)|$ is majorized by a constant multiple of $\gamma(t)h(t)$. But (11) also shows that $h(t)$ is positive for all t if the trivial solution, $x(t) \equiv 0$, of (2) is excluded. Hence, the preceding inequality, when combined with the assumption (18), shows that

$$\int_{\Sigma} |\, d \log h(t)\, | < \infty \tag{20}$$

where Σ denotes the sequence of all t-intervals which consist of the t-values of the second kind.

It is clear from (20), where $h(t) > 0$, that $\log h(t)$ tends to a finite limit, hence $h(t)$ to a finite positive limit, if t tends to ∞ so as to be confined to the set, Σ, of the t-values of the second kind. On the other hand, if t tends to ∞ on the set of the t-values of the first kind, then $h'(t) \leqq 0$, hence $h(t)$ must then tend either to a finite limit or to $-\infty$. But the second case is impossible, since $h(t) > 0$, by (11). Accordingly, $h(t)$ stays bounded whether t tends to ∞ on Σ or on the complement of Σ. In view of (11), this proves that $x(t)$ and $x'(t)$ are bounded as $t \to \infty$, as claimed by (i bis).

7. It is seen from (19) that the criterion (18) for the stability of (2) can be thought of as the analogue of the criterion (3) for the stability of (1) (except that, for (2), the strict positive definiteness of $-F(t)$ is assumed in (i bis), in order to complement the strict positive definiteness of the kinetic term $|x'|^2$, of (6), the Hamiltonian of (2)).

The traditional connection between (1) and (2) starts with (2), and, if n is the number of components in (2), it replaces n, x and F by $2n$, (x, x') and

$$A = \begin{bmatrix} 0 & I \\ F & 0 \end{bmatrix} \tag{21}$$

[9]For a slightly weaker version, cf. the reference[5].

[10]A. Wintner, Amer. Journ. Math. 69 (1947) 5-13.

respectively (I denotes the n-rowed unit matrix). But there is also another formal connection between (1) and (2), a connection which starts with (1).

Professor W. Prager was good enough to call my attention to a paper which he published in a not easily accessible periodical[11] and in which, starting with a scalar equation of the form (2), he is led, by differentiation of (3), to the consideration of scalar equations (of third or fourth order in x) which are of second order in x' or x''. An interpretation of this procedure in the phase space of (2) suggests a corresponding procedure for (1), as it stands.

The result, though straighforward enough, is quite instructive. In fact, it turns out that *what leads from* (1) *to* (2) (*for any* n) *is precisely the* (*matrix*) *equation of Riccati*:

$$F(t) = A'(t) + A^2(t). \tag{22}$$

In fact, if $A(t)$ is differentiable, then (1) implies that $x'' = A'x + Ax'$, and so (2) follows from (1) if F is defined by (22).

Needless to say, the case (22) of (2) is only a necessary condition for a solution $x(t)$ of (1), simply because the general solution of (1) depends on n, whereas that of (2) depends on $2n$, integration constants. The pitfalls of this situation can be illustrated as follows:

Let $F(t)$ be any n by n matrix which is continuous for large positive t and has the property that, when t is fixed, its symmetric part

$$F_*(t) \text{ is non-negative definite} \tag{23}$$

(for instance, that $F(t)$ is symmetric and non-negative). Then it is known[12] that, if $x(t)$ is any solution vector of (2), the graph of $s = |x(t)|^2$ in the (t,s)-plane always turns its convexity toward the t-axis and that, if $x_1(t), \cdots, x_n(t), x_{n+1}(t), \cdots, x_{2n}(t)$ is a complete set of linearly independent solutions of (2), then it is possible to choose $x_1(t), \cdots, x_n(t)$ to be bounded vectors (hence, by convexity, such that $|x_1(t)|^2, \cdots, |x_n(t)|^2$ tend to finite, non-negative limits as $t \to \infty$), whereas $x_{n+1}(t), \cdots, x_{2n}(t)$ are unbounded (which implies that $|x(t)| \to \infty$ holds for any solution vector which is linearly independent of $x_1, \cdots, x_n$).

Let this be applied to the particular case (23). It then follows that, if (22) satisfies (23), the case (22) of (2) has exactly n, linearly independent, solution vectors which are bounded as $t \to \infty$. But none of these n solutions will in general be a solution of (1) also. In fact, if $A(t) = I$ for all t, then (22) satisfies (23), but (1) reduces to $x' = x$ and has, therefore no bounded solution distinct from $x(t) \equiv 0$.

[11]W. Prager, Rev. de la Fac. des Sciences de l'Univ. d'Istanbul, 1 (1936) 37-43.

[12]See the paper referred to under [2].