TWO DIMENSIONAL COMPRESSIBLE SHEAR FLOW IN A CHANNEL*

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1. Introduction. As far as the author is aware, no theoretical or numerical solution has been obtained for the two-dimensional, inviscid, adiabatic rotational flow of a compressible gas through a divergent channel such as shown in Fig. 1. Kramer and Stanitz [2] have obtained a relaxation solution for the incompressible shear flow in a 90° elbow. Mitchell [3] has derived a method for evaluating the non-isentropic rotational field downstream of the bow shock wave formed when a supersonic stream impinges on a blunt nosed obstacle.

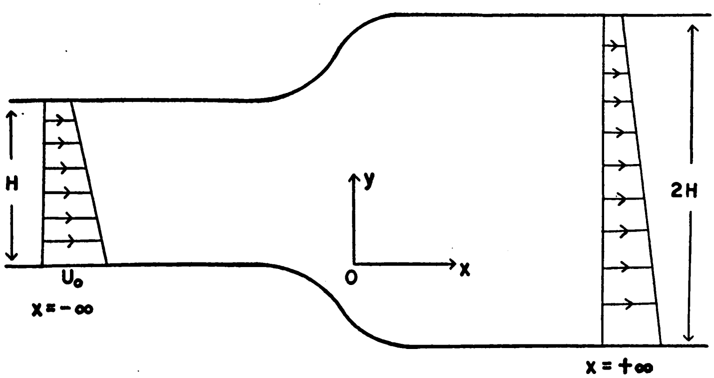

Fig. 1

In the present paper, it is intended to illustrate the principal physical features of the completely sub- and super-sonic adiabatic rotational flow through a channel, with particular reference to the effect of the divergence of the channel on the vorticity and the stream lines.

2. Fundamental equations and boundary conditions. In the two-dimensional steady motion of an inviscid, adiabatic, rotational compressible gas through a divergent channel (see Fig. 1), the stream function ψ is defined in cartesian coordinates by the equations

$$q_x = \rho^{-1} \frac{\partial \psi}{\partial y}, \qquad q_y = -\rho^{-1} \frac{\partial \psi}{\partial x},$$

where q_x and q_y are the cartesian velocity components, and ρ is the density. The vorticity ω is defined by

$$\omega = \frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y}, \tag{1}$$

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which, from Vaszonyi [4], gives the equation for ψ as

$$\frac{\partial}{\partial x}\left(\rho^{-1}\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho^{-1}\frac{\partial\psi}{\partial y}\right)-\rho\frac{\partial h_s}{\partial\psi}=0, \tag{2}$$

where h_s is the stagnation enthalpy.

Bernoulli's equation for an adiabatic gas may be written as

$$\gamma(\gamma-1)^{-1}p/\rho+\tfrac{1}{2}q^2=h_s\,, \tag{3}$$

where p is the pressure, q the velocity, γ the ratio of specific heats, and $p\rho^{-\gamma}=$ constant. In an adiabatic rotational flow, h_s is necessarily a function of ψ.

As to the nature of the boundary conditions necessary for a unique solution to these equations in the general case, there is still some doubt. In the particular problem of the flow through a divergent channel with straight parallel sides far upstream and downstream of the constriction, the boundary conditions are more straightforward. The physical assumption is that at infinity upstream and downstream of the constriction the flow is parallel to the channel walls. Thus, taking the x-axis parallel and along the lower wall in the narrow section of the channel, the condition is $\partial\psi/\partial x=0$ at $x=\pm\infty$, the constriction being $x=0$. Since at $x=\pm\infty$, the flow is parallel, the pressure, and consequently the density, is constant across the channel. Accordingly, if h_s (or the vorticity) is given as a function of ψ, ψ is obtained from Eq. (2), and is thus known round a closed boundary, enclosing the field of flow, which in the elliptic case (subsonic flow) defines a unique solution over the complete field. Alternatively, if $\partial\psi/\partial x=0$, and ψ is given upstream at $x=-\infty$, and $\partial\psi/\partial x=0$ at $x=+\infty$, a solution is again defined.

Letting the suffix zero denote quantities at $x=-\infty$, it is convenient at this stage to introduce non-dimensionalising quantities p_0 , ρ_0 , H the channel width, and c_0 the speed of sound, where $c_0^2=\gamma p_0/\rho_0$. Thus, with only slight change in notation, ψ, p, ρ, q, ω, h_s , x and y, now stand for the non-dimensional quantities $\psi(\rho_0c_0H)^{-1}$, p/p_0 , ρ/ρ_0 , q/c_0 , $\omega H/c_0$, h_s/c_0^2 , x/H and y/H. In the following, it must be remembered that all quantities are non-dimensional.

In the specific problem, let the initial vorticity ω_0 be a constant, which from Eq. (1) gives

$$(q_x)_0=M_0-\omega_0y, \tag{4}$$

where M_0 is the initial Mach number of the flow on the lower channel wall. The density is initially constant and equal to unity, which gives

$$\psi=M_0y-\tfrac{1}{2}\omega_0y^2. \tag{5}$$

From Eqs. (3), (4) and (5) the stagnation enthalpy h_s is given as a function of ψ by

$$h_s=(1/\gamma-1+\tfrac{1}{2}M_0^2)-\omega_0\psi. \tag{6}$$

The function $h_s(\psi)$ given by Eq. (6) remains unaltered throughout the flow, since h_s is constant along a streamline.

In the parallel flow region at $x=+\infty$, where the channel width is $2H$ (the general case rH is treated in a similar manner), $\partial\psi/\partial x=0$. Denoting conditions in this region by the suffix unity, Eq. (2) reduces to

$$\frac{\partial}{\partial y}\left(\rho^{-1}\frac{\partial \psi_1}{\partial y}\right) = -\rho_1\omega_0 ,$$

which on integration gives

$$\psi_1 = A + By - \tfrac{1}{2}\rho_1^2\omega_0 y^2,$$

where A and B are constants. On the lower channel wall at $y = -\frac{1}{2}$, $\psi_1 = 0$, and on the upper wall at $y = 3/2$, $\psi_1 = M_0 - \frac{1}{2}\omega_0$ [from Eq. (5)], which results in

$$\psi_1 = \tfrac{1}{8}(2M_0 - \omega_0 + 3\omega_0\rho_1^2) + \tfrac{1}{4}(2M_0 - \omega_0 + 2\omega_0\rho_1^2)y - \tfrac{1}{2}\omega_0\rho_1^2 y^2. \qquad (7)$$

Thus, with the exception of ρ_1, all quantities are known analytically in the parallel flow region at $x = +\infty$. Equation (3) must be satisfied at all points in the field, which in dimensionless form gives

$$(\gamma - 1)^{-1}\rho_1^{\gamma-1} + \frac{1}{2}\left(\rho_1^{-1}\frac{\partial \psi_1}{\partial y}\right)^2 = (\gamma - 1)^{-1} + \tfrac{1}{2}M_0^2 - \omega_0\psi_1 ,$$

where ψ_1 is given by Eq. (7). This last equation reduces to

$$\begin{aligned}\omega_0^2\rho_1^4 + 2(\gamma - 1)^{-1}\rho_1^{\gamma+1} + \rho_1^2[M_0\omega_0 - \tfrac{1}{2}\omega_0^2 - M_0^2 - 2(\gamma - 1)^{-1}]& \\ + (\tfrac{1}{2}M_0 - \omega_0/4)^2 = 0.&\end{aligned} \qquad (8)$$

By Descarte's theorem, there cannot be more than two real positive roots of Eq. (8). It should be noted that Eq. (8) is not the general equation, but is that for a divergent two-dimensional channel whose area ratio is two. The general equation for any area ratio can be obtained quickly using the above method.

In Eq. (3), the stagnation enthalpy h_s is constant along a stream line (although it varies from one stream line to another), and so for any one such stream line

$$(\gamma - 1)^{-1} + \tfrac{1}{2}q_0^2 = (\gamma - 1)^{-1}\rho_1^{\gamma-1} + \tfrac{1}{2}q_1^2 .$$

From this last equation, it is seen that if $q_1 < q_0$, $\rho_1 > 1$, and if $q_1 > q_0$, $\rho_1 < 1$. Thus, the fact that $\rho_1 > 1$ for subsonic flow, and $\rho_1 < 1$ for supersonic flow is also the case when $\omega_0 > 0$. It is true for $\omega_0 = 0$, which is the well known irrotational flow through a divergent channel. Accordingly, the appropriate solution of Eq. (8) is chosen in each case: these are now considered separately.

3. Subsonic flow. In subsonic flow through the channel, the velocity at station unity is less than at station zero, and so ρ_1 is necessarily greater than unity. Thus, the requisite solution of Eq. (8) is that for $\rho_1 > 1$. With the restriction on ω_0, given subsequently by Eq. (10), it can be quickly shown (by the method of signs) that the other real solution (> 0) of Eq. (8) is that in which $\rho_1 < 1$. From Eqs. (1) and (2), the final vorticity ω_1 is given as

$$\omega_1 = -\rho_1\frac{\partial h_s}{\partial \psi} = \rho_1\omega_0 , \qquad (9)$$

which gives $\omega_1 > \omega_0$. That is, in the rotational flow of a compressible gas through a divergent channel, the vorticity will in general increase in value. Figure 2 illustrates how ω_1 varies with ω_0, for several values of M_0. There is a discontinuity in ω_1, which will occur when $q_0 > 1$ and $q_0 < 1$ over some section of the channel near $x = -\infty$. In such a case the problem is of mixed type, and there is still some doubt as to the boundary

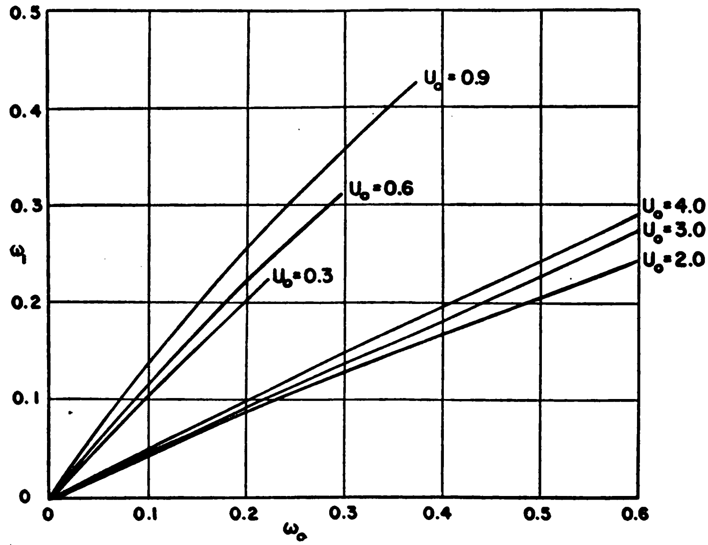

Fig. 2

conditions necessary for a solution to be defined. This difficulty does not arise when q_0 is everywhere greater than or less than unity.

In any physical flow, the velocity in the parallel flow regions must necessarily be in the direction of increasing x, that is, $q > 0$ at all points near $x = \pm\infty$. Thus, ω_0 must satisfy $\omega_0 < M_0$ from Eq. (4), and $\omega_0 < 2M_0\,(1 + 4\rho_1^2)^{-1}$ from Eq. (7). Together these give the following restriction on the initial vorticity ω_0 :

$$\omega_0 < 2M_0(1 + 4\rho_1^2)^{-1} < M_0\,. \tag{10}$$

Equation (10) ensures that there is no 'back' flow region (that is $q > 0$) in the field. The importance of this lies in the fact that if $\omega_0 < M_0$ ensuring that $q > 0$ near $x = -\infty$, the divergence of the channel may be such that the vorticity is increased sufficiently to make $q < 0$ at some points in the region near $x = +\infty$: that is ω_0 violates the first inequality in Eq. (10). Physically this could perhaps result in breakaway of the flow, the cause of which has always been entirely attributed to viscosity effects. It should be pointed out that Eq. (10) is again for an area ratio of two.

The effect of the divergence on the stream lines is marked, and is probably shown most profitably by the final deflection in the central stream line at $x = -\infty$, that is $(\psi_0)_{y=\frac{1}{4}} = (\frac{1}{2}M_0 - \omega_0/8)$. Substitution of this value for ψ_1 in Eq. (7), gives a quadratic equation, and the requisite value of y at $x = +\infty$ as the appropriate solution. The non-dimensional deflection δ of this stream line at $x = +\infty$ is given by Eq. (11), where y is the solution to the quadratic with the negative sign with the root bracket. The deflection δ must tend to zero as ω_0 tends to zero (irrotational flow), which necessitates the negative sign used in Eq. (11). Thus, the deflection is given by

$$\delta = y - 0.5 = (2\omega_0\rho_1^2)^{-1}[(M_0 - \tfrac{1}{2}\omega_0) - (M_0^2 + \omega_0^2/4 + 4\omega_0^2\rho_1^4 - \omega_0 M_0 - \omega_0^2\rho_1^2)^{1/2}], \tag{11}$$

from which it is seen that $\delta \to 0$ as $\omega_0 \to 0$. The root bracket in Eq. (11) may be written as $[(M_0 - \frac{1}{2}\omega_0)^2 + \omega_0^2\rho_1^2(4\rho_1^2 - 1)]^{\frac{1}{2}}$, which is greater than $(M_0 - \frac{1}{2}\omega_0)$ when $\rho_1 > 1$, which is the subsonic case. Thus, from Eq. (11), $\delta < 0$ when the flow is subsonic, for any ω_0 and M_0 satisfying Eq. (10). The variation in δ with initial vorticity for several values of the initial velocity M_0, is illustrated in Fig. 3. One important feature of the flow arising from the figure is the fact that this deflection decreases with increase in the initial velocity for the same value of the vorticity. It should also be noted that the central stream line moves to a region of higher velocity at $x = +\infty$ than the central value in this region.

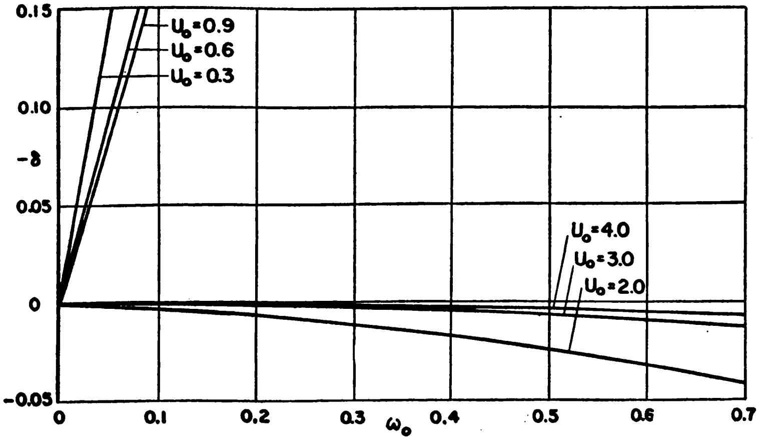

Fig. 3

In a problem of this type, which is elliptic, a simultaneous relaxation process can be carried out over the complete field of flow. The author has in fact evaluated such a problem, where the constriction was discontinuous, each wall having two right-angled corners. This relaxation solution was compared with a similar one for incompressible flow through the same channel, and it was found that in general the displacement effects were smaller in the compressible flow, but of the same sign.

4. Supersonic flow. In the completely supersonic flow through the channel, assuming that no shock waves are present, the differential equations are intrinsically different. This case is a hyperbolic problem, amenable to solution by characteristics, and conditions need only be given initially at $x = -\infty$. However, from physical considerations, it is assumed that the flow near $x = +\infty$ is parallel to the walls of the channel as before.

In this case the divergence of the channel increases the velocity, and the value of ρ_1 is the requisite solution of Eq. (8) satisfying $0 < \rho_1 < 1$. Accordingly, it follows from Eq. (9) that $\omega_1 < \omega_0$, and the restriction on ω_0 for completely supersonic flow at all points is, from Eq. (4)

$$\omega_0 < M_0 - 1. \tag{12}$$

Thus, the divergence decreases the vorticity, and the variation of ω_1 with ω_0 for several values of M_0 is shown in Fig. 2. In this case $q_0 > 1$ at all points.

The deflection in the central stream line is again given by Eq. (11), the negative sign being again used with the root bracket for the same reason as in Sec. 3. From Eq. (8), with ω_0 and M_0 (> 1) satisfying Eq. (12), it can be shown that ρ_1 is in fact less than 0.5 [this can also be seen immediately from Fig. 2 and Eq. (9)] and so the root exponent in Eq. (11) becomes $[(M_0 - \frac{1}{2}\omega_0)^2 - \omega_0^2\rho_1^2(1 - 4\rho_1^2)]^{\frac{1}{2}}$, which, since $1 > 4\rho_1^2$, is less than $(M_0 - \frac{1}{2}\omega_0)$. Thus, Eq. (11) gives $\delta > 0$ for all ω_0 and M_0 (> 1) satisfying Eq. (12) in the completely supersonic flow (that is, where $\rho_1 < 1$). The variation of δ with ω_0 and M_0 is shown in Fig. 3. The essential difference between this case and the subsonic case, is that the central stream line at $x = -\infty$ is deflected towards a region of lower velocity at $x = +\infty$ than the central value at this station. Further, the magnitude of the deflection is very much smaller (in fact an order of magnitude different) in the supersonic case. One point of similarity is that the deflection decreases with increase in velocity.

In supersonic shear flow the effect of the vorticity on the deflection of the downstream stream lines is an order of magnitude different from that in the subsonic flow. This phenomenon substantiates the experimental work of Davies [1], who found that the deflection in the free stream of the stagnation stream line when a pitot tube is placed in a supersonic shear flow is opposite in sign and of a very much smaller magnitude from that obtained by Young and Maas [5] in the low subsonic case.

5. Discussion of results. By considering the differential equations (2) and (3), the variation in the vorticity due to the divergence in the channel is obtained, when the flow is, in one case, completely subsonic and, in the other, completely supersonic. It is shown that the difference between initial vorticities becomes smaller with increase in velocity in both cases, but in the subsonic case the final vorticity is greater than the initial value, the opposite being the case in the supersonic flow, as is shown in Fig. 2.

The effect of the divergence of the channel on the stream lines is characterised by consideration of the deflection in the initial central stream line, as is illustrated in Fig. 3. It is found that the deflection in the supersonic case is of an order of magnitude smaller and opposite in sign, from that found in the subsonic case. This result is in agreement with the state of affairs in the experimental work on supersonic flow past a pitot tube carried out by Davies.

One final remark is that there is a possibility of 'back' flow, which could result in breakaway of the flow in the subsonic rotational flow through a channel of this type. Restrictions on the initial rotation and channel width necessary for no 'back' flow to be present are derived.

The case of a divergent channel with general area ratio can be derived using the method described in this paper, with a proportional increase in algebraic labour.

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