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# ON THE DIFFUSION OF TIDES INTO PERMEABLE ROCK OF FINITE DEPTH*

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**1. Introduction.** It has been observed in the irrigation wells of the Hawaiian Islands that the water-level fluctuations have frequency components corresponding to those of the ocean tides [1]. This phenomenon was analyzed by Carrier and Munk [2], assuming the observed ground-water fluctuations to represent a diffusive transmission of the tidal disturbances through the porous volcanic structure of the island. The purpose of the investigation was to use the results in estimating the permeability of the porous medium.

In [2] it was assumed that the porous medium was infinitely deep. In actual fact, however, there will be an essentially impenetrable bounding surface (see Fig. 1). This paper is concerned with the analysis of the same problem treated in [2], but taking account of the bounding bottom surface. Numerical computations are carried out for several values of the dimensionless depth. Also the limiting case of shallow water theory is studied. Using the results of the infinite depth, shallow depth, and finite depth theory it is possible from the graphs given in Figs. 2 and 3 to estimate the amplitude and phase lag in the fluctuations of the ground-water as a function of the distance inland for various values of the dimensionless depth. It is found that for the values of the physical parameters which are probably of most concern the infinite depth theory gives satisfactory results in the region of interest.

**2. Formulation of the problem.** Although the formulation and the first part of the

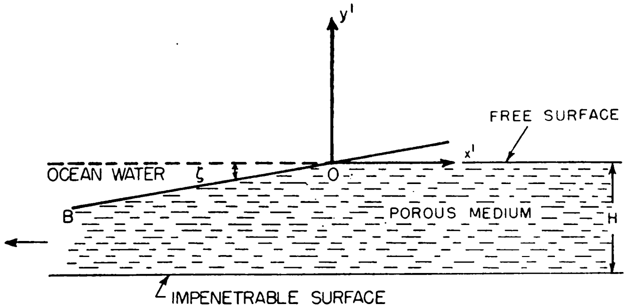


Fig. 1.

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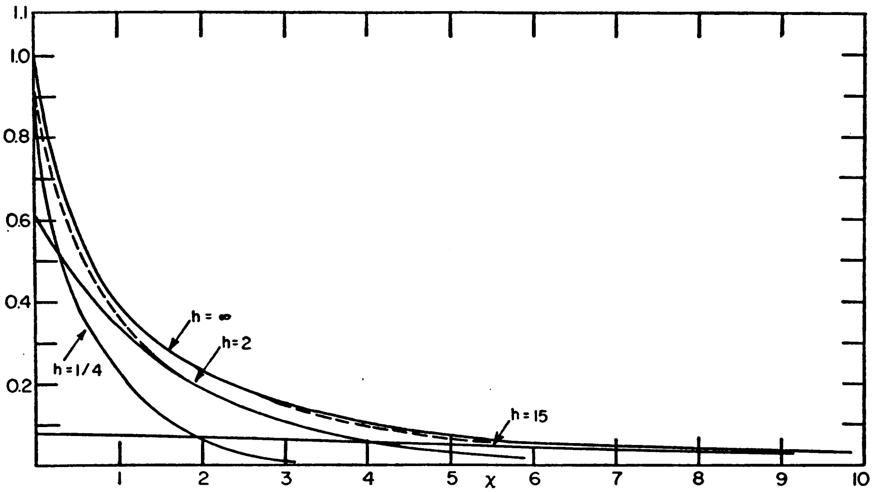


FIG. 2. Ground-water amplitude *vs.* distance inland, $x$. (Dotted lines indicate the expected correction to the computed curves when all modes are considered)

analysis of this problem follows quite closely that related in [2] it is convenient, for the sake of completeness, to repeat part of that work here. As the water in the ocean bounded by $Oy'$ and $OB$ (see Fig. 1) rises and falls about its mean level $OA$, the pressure on the line $OB$ varies. Corresponding to this periodic change in pressure on $OB$ we can expect periodic fluctuations in the free surface of the ground-water, i.e. $Ox'$.

The equations governing the motion of the fluid in the porous medium are the conservation of mass

$$\operatorname{div}(\rho \mathbf{v}) = -\frac{\partial}{\partial t}(\rho\theta), \tag{2.1}$$

and Darcy's law which replaces the conservation of momentum law (see [3]),

$$\mathbf{v} = -(k/\mu) \operatorname{grad}(p - p_0). \tag{2.2}$$

Here $\rho$ and $\mu$ are the density and the viscosity of the fluid; $\theta$ and $k$ are the porosity and the permeability of the material; $\mathbf{v}$ is the velocity with components $u$ and $v$ in the $x'$ and $y'$ direction; $p$ is the gauge pressure; $p_0 = -\rho_0 g y'$ is the pressure when the fluid motion is zero; $\rho_0$ is the mean density; and subscript notation indicates partial differentiation. The simple compressibility law used by Carrier and Munk is

$$\rho\theta = \rho_0\theta_0\{1 + \delta(p - p_0)\}, \tag{2.3}$$

where $\delta$ is essentially $(\rho_0 c^2)^{-1}$ with $c$ the speed of sound in the fluid. It should be pointed out that Eq. (2.2) says that the pressure gradient is proportional to a velocity rather than an acceleration as in the Navier-Stokes equation. As a result we will obtain finally an equation of the diffusion type rather than a wave equation; hence the free surface amplitude will decay in $x'$.

The boundary conditions expressed in terms of the pressure $p$ are

$$p = 0, \quad \text{on the free surface}, \tag{2.4a}$$

$$p = -\rho_0 g y'_{OB} + q_1 e^{i\omega t} \text{ on } OB, \tag{2.4b}$$

$$\partial(p - p_0)/\partial y' = 0 \text{ on } y' = -H, \tag{2.4c}$$

$p = 0$, on the free surface,
where $H$ is the depth of the porous medium. The last boundary condition results from the requirement that the normal component of velocity be zero on the impenetrable bottom. The pressure $q_1$ is of course directly proportional to the tidal-wave amplitude.

If we let $q = p - p_0$, and combine Eqs. (2.1), (2.2) and (2.3) we obtain

$$\Delta q = (\mu\theta_0\delta/k)q_t, \tag{2.5}$$

where $\Delta$ is the Laplacian operator. If we denote by $\eta(x', t)$ the $y'$ coordinate of the free surface, Eq. (2.4a) implies $q(x', \eta, t) = \rho_0 g\eta$, but on the free surface $\eta_t = v/\epsilon$, hence using Eq. (2.2) our boundary condition (2.4a) may be expressed as

$$q_t + (\rho_0 gk/\mu\theta_0)q_{y'} = 0 \quad \text{on} \quad y' = \eta, x > 0. \tag{2.6a}$$

Actually as in the usual linear theory of water waves this boundary condition is to be applied on $y = 0$. The boundary conditions (2.4b) and (2.4c) may be written as

$$q(\text{on } OB) = q_1 e^{i\omega t}, \tag{2.6b}$$

$$q_{y'} = 0, y' = -H. \tag{2.6c}$$

We shall only solve this problem in the case that the line $OB$ occupies the half-line $y = 0, x \leq 0$. That is we take $\zeta = 0°$ (see Fig. 1). Actually this is fairly realistic since $\zeta$ is probably of the order of 5° or so.

Finally if we introduce the following dimensionless variables

$$\left.\begin{aligned} &\tau = \omega t, \qquad x = x'/L, \qquad y = y'/L, \qquad h = H/L, \\ &L = (\rho_0 gk)/(\mu\theta_0\omega), \qquad \epsilon = (\rho_0^2 g^2 k\delta)/(\mu\theta_0\omega), \qquad q = q_1\varphi(x, y)e^{i\omega t}, \end{aligned}\right\} \tag{2.7}$$

we obtain

$$\Delta\varphi - i\epsilon\varphi = 0, \tag{2.8}$$

with the boundary conditions

$$\varphi_y + i\varphi = 0, \qquad y = 0, \qquad x > 0, \tag{2.9a}$$

$$\varphi = 1, \qquad y = 0, \qquad x < 0, \tag{2.9b}$$

$$\varphi_y = 0, \qquad y = -h, \qquad -\infty < x < \infty. \tag{2.9c}$$

The free surface $\eta(x, t)$ is $(\rho_0 g)^{-1} q(x, 0, t)$, but from Eqs. (2.7) $q = q_1\varphi(x, y) \exp(i\omega t)$ so

$$\eta(x, t) = \frac{q_1}{\rho_0 g}\varphi(x, 0)e^{i\omega t} \; x > 0. \tag{2.10}$$

So the problem of determining the free surface is exactly that of determining $\varphi(x, 0)$. The combination of parameters $q_1/\rho_0 g$ is the maximum height of the tidal-wave measured from $y' = 0$.

Before proceeding to a solution of the problem defined by Eqs. (2.8) and (2.9) it is perhaps worthwhile to mention briefly the size of the parameters which appear in this problem. We have $\mu/\rho_0 = 0\,(10^{-2}cm^2/\text{sec})$, $k = 0(5 \times 10^{-6}\text{cm}^2)$, $\theta = 0(.20)$, $g = 980$ cm/sec$^2$, and $\omega$ for a twenty-four hour tide is $2\pi/24$ hours, hence $L = 0(1000 \text{ ft})$. Since

$c$ is $0(5000$ ft/sec) for water, $\epsilon$ is a very small number, $0(10^{-4})$. Finally, a reasonable value for the depth of the ocean is about three miles so $h$ may be as large as 15.

**3. Shallow water theory.** Before considering the general problem given by Eqs. (2.8) and (2.9) let us look at the limiting case in which the depth $H$ is small enough that we can neglect variations in the $y'$ direction, and also set $v \equiv 0$. Then $u(x', t)$ represents an averaged velocity across the section $-H < y' < 0$. If we assume incompressibility, i.e. $\delta = 0$, the conservation of mass equation appropriate to this situation is

$$H\rho_0 u_{x'} = -\rho_0\theta_0\eta_t \,. \tag{3.1}$$

Darcy's equation reduces to

$$u = -(k/\mu)q_{x'} \,. \tag{3.2}$$

Since there is no variation in the $y$ direction our condition that $q = \rho_0 g\eta$ on the free surface must hold throughout the strip $-H < y' < 0$, $x > 0$. Using this and Eqs. (3.1) and (3.2) we obtain

$$q_{x'x'} - (\mu\theta_0/k\rho_0 gH)q_t = 0, \qquad x' > 0. \tag{3.3}$$

The condition that $q = q_1 \exp(i\omega t)$ for $x \leq 0$ is now applied at $x = 0$; consequently we set $q = q_1\varphi(x') \exp(i\omega t)$. Then Eq. (3.3) becomes

$$\varphi_{x'x'} - i(HL)^{-1}\varphi = 0, \qquad x' > 0. \tag{3.4}$$

An appropriate solution of Eq. (3.4) satisfying a finiteness condition at infinity is

$$\varphi(x', t) = \exp[-(i/HL)^{\frac{1}{2}}x'].$$

Hence

$$\eta(x', t) = (q_1/\rho_0 g) \exp\{-x'/(2HL)^{\frac{1}{2}} + i[\omega t - x'/(2HL)^{\frac{1}{2}}]\}, \tag{3.5a}$$

$$= (q_1/\rho_0 g) \exp\{-x/(2h)^{\frac{1}{2}} + i[\omega t - x/(2h)^{\frac{1}{2}}]\}. \tag{3.5b}$$

Actually, in order for this theory to be valid not only must the wave length of the disturbance be large compared to $H$ as in the usual shallow water theory but also $H$ must be small compared to the other natural length scale, $L$, which appears in the problem, i.e. $h$ must be small. This can be seen by an examination of the behaviour of the solution of the general problem. This is done in Sec. 5 where it is found that for $h \leq 1/4$ we can expect the shallow water theory to be quite accurate. The amplitude and phase lag of $\rho_0 g\eta(x, t)/q_1$ are plotted in Figs. 2 and 3 as a function of $x$ for $h = 1/4$.

**4. Solution of the problem.** To solve the problem defined by Eqs. (2.8) and (2.9) we shall use the method of Fourier transforms and the Wiener-Hopf technique. Let

$$\Phi(\xi, y) = \int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x, y)\, dx. \tag{4.1}$$

Then the transform of Eq. (2.8) is

$$\Phi_{yy} - (\xi^2 + i\epsilon)\Phi = 0. \tag{4.2}$$

A solution of this equation satisfying the boundary condition (2.9c) is

$$\Phi(\xi, y) = A(\xi) \cosh\{(y + h)C\}, \tag{4.3}$$

where $C = (\xi^2 + i\epsilon)^{1/2}$ and $A(\xi)$ is to be determined by satisfying the remaining boundary conditions.

Let

$$g_1(x) = \begin{cases} \lim_{a\to 0} e^{ax} & x < 0 \\ 0 & x > 0 \end{cases}, \tag{4.4a}$$

$$g_2(x) = \begin{cases} 0 & x < 0 \\ \varphi(x, 0) & x > 0 \end{cases}, \tag{4.4b}$$

and

$$f(x) = \varphi_y(x, 0) + i\varphi(x, 0). \tag{4.5}$$

It is clear that $\varphi(x, 0) = g_1(x) + g_2(x)$; hence*

$$\left.\begin{aligned} \Phi(\xi, 0) = A(\xi) \cosh Ch &= G_1(\xi) + G_2(\xi), \\ &= (a - i\xi)^{-1} + G_2(\xi). \end{aligned}\right\} \tag{4.6}$$

Also using Eq. (4.3)

$$F(\xi) = (C \sinh Ch + i \cosh Ch)A(\xi). \tag{4.7}$$

Combining Eqs. (4.6) and (4.7) we obtain

$$F(\xi) = K(\xi)\{G_1(\xi) + G_2(\xi)\}, \tag{4.8}$$

where

$$K(\xi) = \frac{C \sinh Ch + i \cosh Ch}{\cosh Ch}. \tag{4.9}$$

If we recall from Sec. 2 that we wish to determine $\varphi(x, 0)$ it is clear from Eq. (4.4) that our problem is now that of determining $g_2(x)$ and hence $G_2(\xi)$. To determine $G_2(\xi)$ using Eq. (4.8) we shall use the Wiener-Hopf technique. This technique has been used to treat similar problems (see for instance [2, 4, 5]); consequently the analysis will only be briefly outlined here. First $G_1(\xi)$ is analytic in the upper half plane, (*UHP*), $Im(\xi) > -a$; $G_2(\xi)$ is analytic in the *LHP* and $F(\xi)$ is analytic in the *UHP*. The function $K(\xi)$ is analytic and non-vanishing in a strip containing the real axis. This will be seen clearly at the end of this section where $K(\xi)$ is represented as the quotient of two infinite products. It might be noted that though $C = (\xi^2 + i\epsilon)^{1/2}$ is a multivalued function, $K(\xi)$ as defined by Eq. (4.9) consists only of even terms and hence does not have any branch points. Assuming for the moment that we can write $K(\xi)$ as $K_-(\xi)/K_+(\xi)$ where $K_-(\xi)$ is analytic and non-vanishing in the *LHP*, and $K_+(\xi)$ is analytic and non-vanishing in the *UHP* we can rewrite Eq. (4.8) as

$$F(\xi)K_+(\xi) - K_-(-ia)G_1(\xi) = \{K_-(\xi) - K_-(-ia)\}G_1(\xi) + K_-(\xi)G_2(\xi). \tag{4.10}$$

The left hand side of this equation is analytic in the *UHP*, the right hand side is analytic in the *LHP* and they agree in a common strip of analyticity. Hence Eq. (4.10) defines an entire function $E(\xi)$. We shall show shortly that $K_-(\xi) = 0(\xi^{1/2})$ as $|\xi| \to \infty$, $Im(\xi) < 0$ and $K_+(\xi) = 0(\xi^{-1/2})$ as $|\xi| \to \infty$, $Im(\xi) > 0$. Using this and investigating order conditions at infinity we can show that $E(\xi) = 0$, consequently

$$G_2(\xi) = \left\{\frac{K_-(-ia)}{K_-(\xi)} - 1\right\} G_1(\xi). \tag{4.11}$$

*Capital letters are used to denote the Fourier transform.

It is now necessary to determine $K_-(\xi)$ and $K_+(\xi)$. The splitting of $K(\xi)$ is done in a manner exactly analogous to that used by Heins in [4] and [5]. Using the infinite product representation of $\cosh z$, (see [6]) we have

$$M(\xi) = \cosh Ch = \prod_{n=0}^{\infty} \{1 + (2Ch)^2/(2n+1)^2\pi^2\} = m(\xi)m(-\xi), \tag{4.12}$$

where

$$\begin{aligned} m(\xi) = \prod_{n=0}^{\infty} \{&[1 + (4i\epsilon h^2)/(2n+1)^2\pi^2]^{\frac{1}{2}} \\ &+ i2h\xi/(2n+1)\pi\} \exp[-i2h\xi/(2n+1)\pi]. \end{aligned} \tag{4.13}$$

We have inserted the exponentials to insure absolute convergence of the infinite products defining $m(\xi)$ and $m(-\xi)$ in the $UHP$ and $LHP$ respectively. If we write $M(\xi)$ as $M_-(\xi)/M_+(\xi)$ it is clear that $M_-(\xi) = m(\xi)$ has no zeros or poles in the $LHP$. Similarly $1/M_+(\xi) = m(-\xi)$ has no zeros or poles in the $UHP$.

The function $L(\xi) = C \sinh Ch + i \cosh Ch$ has zeros at $Ch = \pm i\beta_n$, $n = 0, 1, 2, \cdots$, where the $\beta_n$ are complex numbers lying in the first quadrant. For $n$ large they may be determined by the asymptotic relation $\beta_n = n\pi + ih/n\pi + 0\ ([n\pi]^2)$. We may write $L(\xi)$ as

$$L(\xi) = i \prod_{n=0}^{\infty} \{1 + (Ch)^2/\beta_n^2\} = il(\xi)l(-\xi), \tag{4.14}$$

where

$$\begin{aligned} l(\xi) = \{[1 + (i\epsilon h^2)/\beta_0^2]^{\frac{1}{2}} + i\xi h/\beta_0\} \prod_{n=1}^{\infty} \{&[1 + (i\epsilon h^2)/\beta_n^2]^{\frac{1}{2}} + i\xi h/\beta_n\} \\ &\exp(-i\xi h/n\pi). \end{aligned} \tag{4.15}$$

Again we have inserted the exponentials in order to insure absolute convergence in the appropriate half planes. If we write $L(\xi)$ as $L_-(\xi)/L_+(\xi)$ and take $L_-(\xi) = l(\xi)$, $1/L_+(\xi) = il(-\xi)$ it is clear that $L_-(\xi)$ is free of zeros and poles in the $LHP$ and $L_+(\xi)$ is free of zeros and poles in the $UHP$.

Consequently we have

$$K_-(\xi) = \exp\{\chi(\xi)\}L_-(\xi)/M_-(\xi) = \exp\{\chi(\xi)\}l(\xi)/m(\xi), \tag{4.16a}$$

$$K_+(\xi) = \exp\{\chi(\xi)\}L_+(\xi)/M_+(\xi) = \exp\{\chi(\xi)\}m(-\xi)/il(-\xi). \tag{4.16b}$$

We shall choose the factor $\exp\{\chi(\xi)\}$ introduced in Eqs. (4.16a) and (4.16b) in such a manner that $K_-(\xi)$ and $K_+(\xi)$ have algebraic behaviour as $|\xi| \to \infty$ in the $LHP$ and $UHP$ respectively.

To investigate the behaviour of $K_-(\xi)$ for $Im(\xi) < 0$, $|\xi| \to \infty$ we first note that the terms involving $\epsilon$ may be neglected against unity for $|\xi| \to \infty$. Since $\beta_n \to n\pi$ as $n \to \infty$ we have that $K_-(\xi)$ is of the order

$$\exp\{\chi(\xi)\}(1 + i\xi h/\beta_0) \prod_{n=1}^{\infty} \{(1 + w/n)\exp(-w/n)\} \Big/$$

$$\prod_{n=1}^{\infty} \{[1 + 2w/(2n+1)]\exp[-2w/(2n+1)]\},$$

where $w = i\xi h/\pi$. Now using the relation that

$$1/\Gamma(w) = we^{\gamma w} \prod_{n=1}^{\infty} (1 + w/n)e^{-w/n},$$

and Stirling's asymptotic formula for the gamma function, (see [6]) we obtain

$$K_-(\xi) = 0\{w^{\frac{1}{2}} \exp [\chi(\xi) + w ln 4]\},$$

for $Im(\xi) < 0$, $|\xi| \to \infty$. So choosing $\chi(\xi) = -w \ln 4 = -(i\xi h \ln 4)/\pi$ we have that $K_-(\xi) = 0(\xi^{1/2})$ as $|\xi| \to \infty$, $Im(\xi) < 0$. A similar argument will show that $K_+(\xi) = 0(w^{-1/2})$ for $|\xi| \to \infty$, $Im(\xi) > 0$. With these order relations it is not difficult to show that $E(\xi)$ is zero as mentioned earlier; and hence we obtain Eq. (4.11) for $G_2(\xi)$ where $K_-(\xi)$ is defined by Eq. (4.16a). In particular it can be seen from Eq. (4.16a) and the definitions of $1(\xi)$ and $m(\xi)$ that $K_-(0) = 1$.

Using the usual inversion formula we have that

$$G_2(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\xi x}\{[K_-(-ia) - K_-(\xi)]/(a - i\xi)K_-(\xi)\}\, d\xi, \tag{4.17a}$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\xi x} \frac{K_-(-ia) \cosh Ch - K_+(\xi)K(\xi)}{(a - i\xi)K_+(\xi)K(\xi)}\, d\xi. \tag{4.17b}$$

In the limit as $a \to 0$ it is clear from Eq. (4.17a), that $G_2(\xi)$ is not singular at the origin; hence we may actually take the real axis as our path of integration in evaluating $g_2(x)$. Of course we shall actually close our path of integration in the *UHP* when $x > 0$ and in the *LHP* when $x < 0$. For the case $x > 0$ it is convenient to use Eq. (4.17b) for evaluating $g_2(x)$. Since $K_+(\xi)$ is non-vanishing in the *UHP*, $g_2(x)$ will be simply the sum of the residues at the poles of the integrand which occur at $Ch = i\beta_n$, $n = 0, 1, \cdots$. Carrying out this straightforward computation we obtain in the limit

$$g_2(x) = h^{-1} \sum_{n=0}^{\infty} \left(\frac{\beta_n}{\alpha_n}\right)^2 \frac{e^{-\alpha_n x}}{(\beta_n^2 + ih - h^2)K_+(i\alpha_n)}, \tag{4.18}$$

where $\alpha_n = (\beta_n^2/h^2 + i\epsilon)^{1/2}$. In the particular case that the fluid is incompressible, i.e. $\epsilon = 0$, $\beta_n/\alpha_n = h$ and Eq. (4.18) becomes

$$g_2(x) = h \sum_{n=0}^{\infty} \frac{e^{-\alpha_n x}}{(h^2\alpha_n^2 + ih - h^2)K_+(i\alpha_n)}, \qquad \alpha_n = \beta_n/h. \tag{4.19}$$

**5. Numerical computations and discussion.** In this section we shall only be concerned with the case in which the fluid may be considered as incompressible, then $g_2(x)$ is given by Eq. (4.19). In [2] a few values of $g_2(x)$ were computed for $\epsilon = .01$ and compared to the $\epsilon = 0$ case; the amplitude and phase lag in the ground-water fluctuation for $\epsilon = .01$ were slightly lower than for $\epsilon = 0$. Since $\epsilon$ is however $0(10^{-4})$ we should expect very little error in actually setting $\epsilon = 0$.

First let us determine when the shallow water theory solution given by Eqs. (3.5) may be expected to be valid. In order for $g_2(x)$ as given by Eq. (4.19) to agree with the shallow water solution it is necessary that $\alpha_0 \sim (i/h)^{1/2}$ as $h \to 0$, and also that all the coefficients of the higher order terms[1] must approach zero. It can be shown with little

[1]We shall refer to the term $\exp(-\alpha_0 x)$ in Eq. (4.19), which is the dominating term as $x \to \infty$, as the fundamental term.

difficulty from an investigation of the transcendental equation $C \sinh Ch + i \cosh Ch = 0$ that for small $h$, $\beta_0 \sim (ih)^{1/2}$ and hence $\alpha_0 \sim (i/h)^{1/2}$. Also upon noting that for small $h$, $\beta_n \sim n\pi$ for $n \geq 1$, it can be seen from Eq. (4.19) with the aid of the representation of $K_+(i\alpha_n)$ given in Eq. (5.1) that all the coefficients of the higher order terms do approach zero as $h \to 0$. Hence $g_2(x)$ as given by Eq. (4.19) does approach the shallow water solution as $h \to 0$. To determine quantitatively when Eq. (3.5b) is valid we have computed $\beta_0$ and $\alpha_0$ as a function of $h$. Also the ratios of the wave length predicted by the shallow water theory, $\lambda = 2\pi/(2HL)^{1/2}$, to $H$ and that of the fundamental mode, $\lambda_0 = 2\pi/Im(\alpha_0)$, to $H$ have been computed. These results are given in Table 1 and

TABLE 1.

*$h$ is the dimensionless depth, $\lambda = 2\pi(2HL)^{1/2}$ is the wave length for shallow water theory, $\lambda_0 = 2\pi L/Im(\alpha_0)$ is the wave length of the fundamental mode for finite bottom theory.*

| $h$ | $\beta_0$ | $\alpha_0(\epsilon = 0)$ | $\lambda/H$ | $\lambda_0/H$ |
|---|---|---|---|---|
| $\to 0$ | $\to (ih)^{1/2}$ | | | |
| .25 | $.3676 + i\ .3382$ | $1.4704 + i\ 1.3528$ | 17.76 | 18.57 |
| .50 | $.5376 + i\ .4548$ | $1.0752 + i\ .9096$ | 12.57 | 13.81 |
| 1.00 | $.8004 + i\ .5702$ | $.8004 + i\ .5702$ | 8.88 | 11.09 |
| 2.00 | $1.1828 + i\ .5832$ | $.5914 + i\ .2916$ | 6.28 | 10.77 |
| 3.00 | $1.3739 + i\ .4775$ | $.4579 + i\ .1592$ | 4.85 | 13.16 |
| 5.00 | $1.5033 + i\ .3090$ | $.3007 + i\ .0618$ | 3.97 | 20.03 |
| 10.00 | $1.5547 + i\ .1569$ | $.1555 + i\ .0157$ | 2.81 | 40.04 |
| 15.00 | $1.5638 + i\ .1046$ | $.1043 + i\ .0070$ | 2.29 | 59.83 |

graphically in Fig. 4. It appears from Fig. 4 that we may expect the shallow water theory to be accurate over the entire range of $x$ for $h \leq 1/4$.

In order to compute $g_2(x)$ for various values of $h$ it is necessary to cast $K_+(i\alpha_n)$ into a form more suitable for numerical analysis than that given by Eq. (4.16a). This can be done in a straightforward manner by using the infinite product representation of the gamma function. We obtain, when $\epsilon = 0$, that

$$K_+(i\alpha_n) = \frac{\beta_n[\Gamma(\beta_n/\pi)]^2 \exp[(\beta_n \ln 4)/\pi]}{2\pi i(1 + \beta_n/\beta_0)\Gamma(2\beta_n/\pi)} \left\{ \prod_{m=1}^{\infty} [(1 + \beta_n/\beta_m)/(1 + \beta_n/m\pi)] \right\}^{-1}. \quad (5.1)$$

In any actual numerical computation the infinite product in Eq. (5.1) is, of course, to be replaced by a finite number of terms (recall that $\beta_m \to m\pi$ as $m \to \infty$). The number of terms that is required to give an accurate answer is of course dependent upon $h$ and $\beta_n$.

TABLE 2.

| $n$ | $\beta_n(h = 2)$ | $\beta_n(h = 15)$ |
|---|---|---|
| 0 | $1.1828 + i\ .5832$ | $1.5638 + i\ .1046$ |
| 1 | $3.3106 + i\ .6499$ | $4.6886 + i\ .3234$ |
| 2 | $6.3014 + i\ .3277$ | $7.8016 +\ .5749$ |
| 3 | $9.4248 + i\ .2138$ | $10.8705 + i\ .9070$ |
| 4 | | $13.7139 + i\ 1.3629$ |
| 5 | | $16.1332 + i\ 1.4236$ |
| 6 | | $18.9644 + i\ 1.0572$ |

In this paper $g_2(x)$ was computed for $h = 2$ and $h = 15$. Values of $\beta_n$ for $h = 2$ and $h = 15$ are given in Table 2. Let us first consider the case $h = 2$. Then the real parts of $\alpha_0$ and $\alpha_1$ are given by .59 and 1.65 respectively hence we may expect the fundamental term to give an accurate result for $x \geq 2$. Carrying out the necessary computations we obtain

$$\frac{\rho_0 g}{q_1}\eta(x, t) = .61 \exp(-.59x) \exp[i(\omega t - .885 - .292x)] + 0[\exp(-\alpha_1 x)]; \qquad h = 2. \tag{5.2}$$

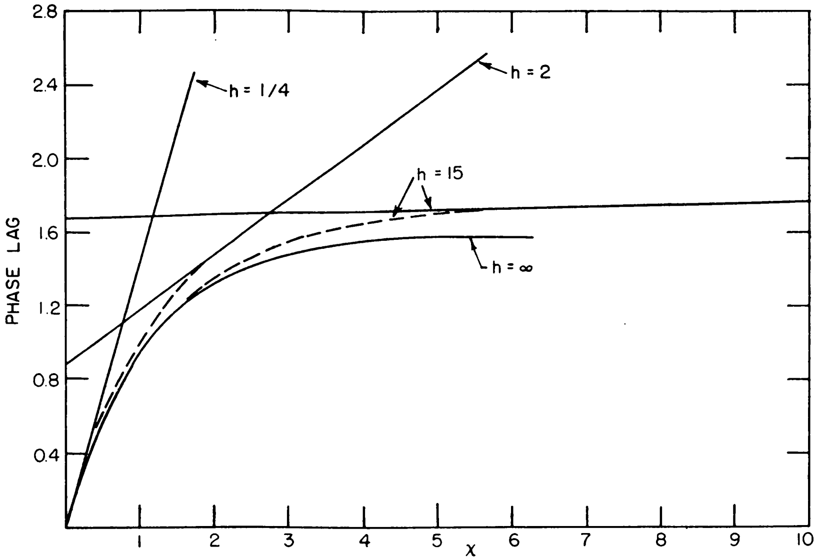


FIG. 3. Ground-water phase lag *vs.* distance inland, $x$. (Dotted lines indicate the expected correction to the computed curves when all modes are considered)

In Figs. 2 and 3 the amplitude and phase lag in the ground-water fluctuation have been plotted. The extrapolation of the results to $x = 0$ are indicated by dotted lines. Actually it is not difficult to obtain another term in the series, but unless particular quantitative information is desired for this value of $h$ it hardly seems necessary to do that. It might be mentioned that three terms were more than sufficient in evaluating the infinite product in Eq. (5.1).

In the case that $h = 15$, the fundamental term can only be expected to be accurate for $x \geq 7$. We obtain

$$\frac{\rho_0 g}{q_1}\eta(x, t) = .074 \exp(-.104x) \exp[i(\omega t - 1.68 - .007x)] + 0[\exp(-\alpha_1 x)]; \qquad h = 15. \tag{5.3}$$

In order to obtain results valid for $x = 1$ or 2 when $h = 15$ would probably require the computation of three or four terms of the series. However in view of the results for $h = 2$ and these results for $x \geq 7$ it is clear that the amplitude curve for $h = 15$ will lie almost exactly on the curve given by the infinite depth theory[2] (see Fig. 2).

[2]The amplitude and phase lag curves for $h = \infty$ have been taken from the results given in [2].

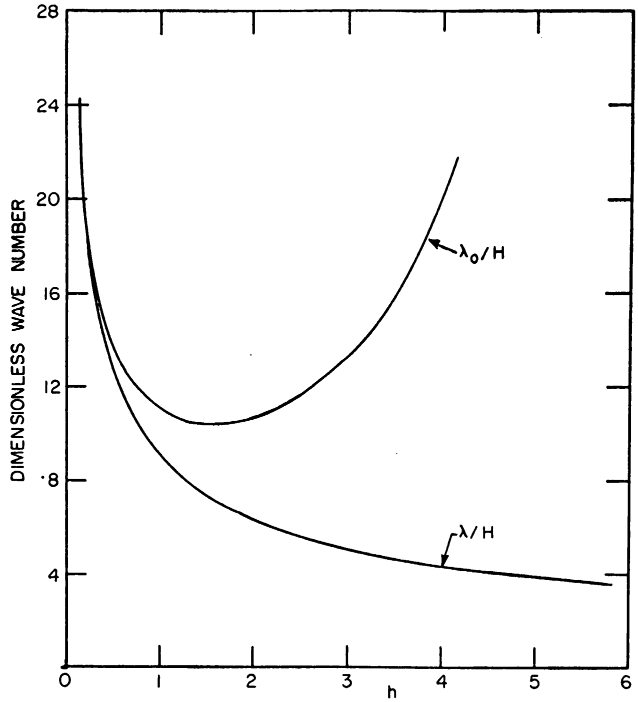


**Fig. 4.** Dimensionless wave number *vs.* $h$; $\lambda$ is the wave length predicted by shallow water theory; $\lambda_0$ is the wave length of the fundamental mode predicted by finite bottom theory.

It is interesting to note that the shallow water theory and finite depth theory predict an exponential decay in $x$ for the amplitude of the ground-water fluctuation; this is in contrast to the algebraic decay, (like $x^{-1}$), predicted by the infinite depth theory. Also the phase lag predicted by the shallow water theory and finite depth theory continues to increase with $x$, while that predicted by the infinite depth theory approaches $\pi/2$ with increasing $x$. (This is clearly illustrated in Fig. 3). That $g_2(x)$ as given by Eq. (4.19) approaches the infinite depth result given in [2] as $h \to \infty$, cannot be seen easily from (4.19). However an examination of $K(\xi)$ as given by Eq. (4.9) shows that as $Ch \to \infty$, $K(\xi) \to i + (\xi^2 + i\epsilon)^{1/2}$ which we might denote by $K_\infty(\xi)$. This function is the one that occurs in [2]. It is interesting to note that $K_\infty(\xi)$ has singularities of the branch point type, and in the limit as $\epsilon \to 0$ these singularities will occur at the origin. This explains the algebraic behaviour of $\eta(x, t)$ for $h = \infty$. In contrast for any finite value of $h$ the strip of analyticity of $K(\xi)$ is finite even when $\epsilon \to 0$, and its singularities are poles rather than branch points; hence the exponential sort of behaviour for $\eta(x, t)$ for finite $h$.

A plausible physical explanation for the fact that the amplitude curves for the ground-water fluctuation lie continuously below one another as $h$ decreases (see Fig. 2) is the following. Imagine that our porous medium and fluid occupy the strip $-H < y < 0$, $-\infty < x < \infty$. Suppose that we apply a uniform pressure on the half line $y = 0$, $-\infty < x < 0$; then fluid in the left half strip will be forced through the gap $-h < y < 0$ and the free surface given originally by $y = 0$, $x \geq 0$ will rise. The amount of fluid that can be forced through this gap, and hence the effect that the pressure variation can have on the free surface, is proportional to the gap distance, $h$. So with decreasing $h$ the amplitude of the free surface fluctuation is lower and dies out more quickly.

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