—NOTES—

ON LINEAR PERTURBATIONS*

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If $f(t)$ and $g(t)$ are continuous functions for large positive t, and if the solutions of the differential equation

$$x'' + f(t)x = 0 \tag{1}$$

are known, how "small" must be the difference $f(t) - g(t)$ (for large t) in order that the general solution of the differential equation

$$y'' + g(t)y = 0 \tag{2}$$

can be guaranteed to have the same asymptotic behavior as the general solution of (1)? The literature consulted answers this question only under assumptions which restrict (1) by certain conditions of stability.[1] No such assumptions are made in the following theorem (which, under stability assumptions, reduces to known results):

With reference to the coefficient function $f(t)$ and a pair $x = u(t)$; $x = v(t)$ of linearly independent solutions of (1), *let the coefficient function $g(t)$ of* (2) *satisfy the following condition*:

$$\int^{\infty} | f - g | (| u |^2 + | v |^2)\, dt < \infty. \tag{3}$$

Then every solution $y = y(t)$ of (2) *is of the form*

$$y(t) = c_1u(t) + c_2v(t) + o(| u(t) | + | v(t) |), \tag{4}$$

where c_1, c_2 are integration constants (which can be chosen arbitrarily). In addition, the asymptotic relation (4) *remains true on differentiation, that is*

$$y'(t) = c_1u'(t) + c_2v'(t) + o(| u'(t) | + | v'(t) |), \tag{5}$$

where $' = d/dt$. The o symbol in $h(t) = j(t) + o(| k(t)|)$ means that $k \neq 0$ for large t and that $(h - j)/k \to 0$ as $t \to \infty$.

The relation (4) reduces to $y \sim c_1u + c_2v$ as $t \to \infty$ if (1) is stable in the sense that $\limsup | x(t)| < \infty$, where $t \to \infty$, holds for all solutions of (1). In this case, condition (3) is certainly satisfied if

$$\int^{\infty} | f - g |\, dt < \infty. \tag{6}$$

But the converse conclusion cannot be made (not even $\limsup | x(t)| < \infty$ is assumed for $x = u$ and $x = v$), since, when $u(t)$ and $v(t)$ happen to be "small" (for large t), then condition (3) requires of the "size" of the perturbation $f - g$ substantially less than what is required by the standard assumption (6).

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[1]In this regard, see A. Wintner, Quart. Appl. Math. **13**, 192-195 (Secs. 2 and 5) (1955).

What is more, the general theorem applies without any stability restriction of the given problem (1). In fact, no matter what the coefficient and the general solution of (1) (that is, the function f and two, linearly independent, solutions u, v) may be, the perturbation $f - g$ of (2) on (1) can always be chosen so small as to satisfy condition (3), that is, so as to bring (2) within the range of the theorem.

The proof of the theorem will consist of two steps.

First, recourse will be had to an elementary lemma which goes back to Bôcher[2] and which, in the binary case at hand, runs as follows: If the coefficients of a homogeneous, linear differential system

$$p' = a_{11}(t)p + a_{12}(t)q, \qquad q' = a_{21}(t)p + a_{22}(t)q \tag{7}$$

are, for large positive t, continuous functions satisfying

$$\int^{\infty} |\, a_{ik}(t) \,|\, dt < \infty, \quad \text{where} \quad i = 1, 2 \quad \text{and} \quad k = 1, 2, \tag{8}$$

then, corresponding to any pair c_1, c_2 of integration constants, the system (7) possesses a unique solution (p, q) satisfying

$$p(t) \to c_1\,, \qquad q(t) \to c_2 \quad \text{as} \quad t \to \infty\,. \tag{9}$$

Next, the following rule of Lagrangian "variation of constants" (a rule which, being purely formal in nature, requires only the continuity of $f(t)$ and $g(t)$ on a t-interval) will be needed.[3] Let two solutions, $x = u$ and $x = v$, of (1) be so chosen that their Wronskian $u(t)v'(t) - v(t)u'(t)$ (which is always a non-vanishing constant) becomes the constant 1, and define, in terms of the difference of the coefficient functions of (1) and (2), a binary matrix function $\| a_{ik}(t) \|$ as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (f - g)\begin{pmatrix} -uv & -v^2 \\ u^2 & uv \end{pmatrix}. \tag{10}$$

Then $y(t)$ is a solution of (2) if and only if there belongs to it a solution (p, q) of the case (10) of (7) in such a way that

$$y(t) = u(t)p(t) + v(t)q(t) \tag{11}$$

becomes an identity.

In addition, this transformation of (2) into (1) is a "contact transformation," in the sense that the following differentiation rule holds for (11):

$$y'(t) = u'(t)p(t) + v'(t)q(t) \tag{12}$$

(in other words, $up' + vq'$ vanishes for all t)[4].

In order to combine this Lagrangian rule with Bôcher's lemma, note that the case (10) of the four conditions (8) is equivalent to the three conditions

$$\int^{\infty} |\, f - g \,|\, |\, w \,|\, dt < \infty, \quad \text{where} \quad w = u^2, uv, v^2,$$

[2]For references, and for certain refinements, see A. Wintner, Am. J. Math. **76**, 183-190 (1954).

[3]For a verification (and for a similar application) of this rule, see A. Wintner, Am. J. Math. **69**, 262-263 (1947).

[4]See formula (34) in the preceding reference[3].

and that, since $2 \mid uv \mid \leqq \mid u \mid^2 + \mid v \mid^2$, the latter three conditions are equivalent to the single condition (3). Accordingly, (3) assures the validity of the limit relations (9) for the general solution (p, q) of the case (10) of (7). But (11) reduces to (4), and (11) to (5), by virtue of (9).

This completes the proof of the theorem. Its assumption (3) is independent of the choice of the two, linearly independent, solutions $x = u(t)$, $x = v(t)$ of (1) which occur in (3). For, on the one hand, $u(t)$ and $v(t)$ cannot vanish at the same t and, on the other hand, the ratio of $\mid u^*(t) \mid^2 + \mid v^*(t) \mid^2$ to $\mid u(t) \mid^2 + \mid v(t) \mid^2$ stays between two positive constant bounds as $t \to \infty$. This is clear from the fact that $u^*(t) = au(t) + bv(t)$ and $v^*(t) = cu(t) + dv(t)$, where a, b, c, d are constants of non-vanishing determinant $ad - bc$.

A particular, but interesting, case of the theorem results if condition (3) is strengthened so as to make possible the elimination of the solution pair (u, v) occurring in (3). Such an elimination is made possible by using a well-known estimate, which was applied in more general forms by Liapounoff[5] and others (L. Schlesinger, G. D. Birkhoff, O. Perron)[6] and which, when particularized to the case of (1), states that

$$\mid x(t) \mid < \text{const. } \exp \left\{ \tfrac{1}{2} \int^t \mid f(s) - 1 \mid ds \right\} \tag{13}$$

holds for every solution $x(t)$ of (1). What then results avoids the implicit hypothesis of the theorem, namely, that (1) has already been solved. In fact, the resulting corollary of the theorem can be formulated as follows.

If the coefficient function $g(t)$ *of* (2) *is so "close" (for large* t*) to the coefficient function* $f(t)$ *of* (1) *that*

$$\int^\infty \mid g(t) - f(t) \mid \exp \left\{ \int^t \mid f(s) - 1 \mid ds \right\} dt < \infty, \tag{14}$$

then the asymptotic behavior of all solutions $y(t)$ *of* (2) *and of their derivatives* $y'(t)$ *is given by* (4) *and* (5), *where* c_1, c_2 *is an arbitrary pair of integration constants and* $u(t)$, $v(t)$ *is a pair of linearly independent solutions* $x(t)$ *of* (1).

In fact, if (13) is applied to $x = u$ and $x = v$, then (3) reduces to (14).

A LEBEDEV TRANSFORM AND THE "BAFFLE" PROBLEM*

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1. Introduction. This note is concerned with the application of the Lebedev transform to what we term the "baffle problem," i.e. the problem of sound radiated by a vibrating circular disk in an infinite rigid baffle. The solution of this problem is not new. It has been solved by Sommerfeld [1] in terms of cylindrical waves, using the Hankel transform, and by Bouwkamp [2] and others in terms of spheroidal waves, using series representation. However, the Lebedev transform offers an equally straightforward method, representing the radiation in terms of spherical waves, and moreover is in

[5] E. Picard, *Traité d'Analyse*, 3rd ed., vol. 3, 1928, p. 385.

[6] For the particular case (13) of the general theorem, see N. Levinson, Duke Math. J. 8, 2-3(1941).

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