ON THE PERIODIC SOLUTIONS OF THE FORCED OSCILLATOR EQUATION*

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Introduction. The phenomenon of "subharmonic vibrations" has been considered by many authors. The research on the subject goes back nearly one hundred years, beginning (probably) with Melde [14]†, Helmholtz [5], and Lord Rayleigh [17], all before the turn of the century, and continuing uninterruptedly to the present. For a fairly complete bibliography on this subject, the reader is referred to the classical paper on non-linear engineering problems by von Karman [8], and the books by Stoker [20], Minorski [15] and McLachlan [13].

In this paper, we shall discuss a single degree of freedom system whose mechanical model might be a mass under the action of an "elastic force" (linear or non-linear, restoring and/or exciting) and of a simple harmonic forcing function of frequency ω. Moreover, we assume that small quantities of positive damping are present in the physical system, but absent from the equation of motion. Such damping can be shown, in general, to reduce the free vibrations to negligibly small amplitudes in a finite time. Consequently, the periodic solutions of this system are those usually referred to as "steady state vibrations." We are assuming here that the effect of small damping on the motion is slight, but later on we shall prove this to be the case.

It follows that the equation of motion of the system is the oscillator equation

$$x^{\cdot\cdot} + f(x) = P_0 \cos \omega t, \tag{1}$$

where P_0 and ω are non-vanishing, finite, real constants, x is a dependent real variable and t is the independent variable time. For the present, the function $f(x)$ remains undefined.

Equation (1) and its harmonic and subharmonic solutions have been explored widely and by a considerable variety of methods, but it seems that the treatment was restricted in most cases to almost linear, and odd, $f(x)$. Ludeke [12] is one of the early investigators who has considered some cases, and experimented with some models, in which the departure of the elastic force from linearity is significant. However, he only considers very special cases; for instance, his treatment of the 1/3 order subharmonic vibration is restricted to special values of the forcing frequency. Moreover, his experimental model may be subject to an equation with periodic coefficients rather than one of the type (1) because the inertia of his pendulum varies harmonically with time. McLachlan [13] has given an example of a special equation of the type (1) in which $f(x)$ is strongly non-linear, and this equation was also given by Cartwright and Littlewood [1]. Moreover, there is a great deal of significant literature [6, 7, 9, 11, 16] exploring solutions of equations like (1). In an earlier paper [19] the origin of subharmonics of odd orders was discussed; however, that investigation contained many conjectures, and the treatment was re-

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†Numbers in square brackets refer to the bibliography at the end of the paper

stricted to odd orders. Here, we would like to lay a proper foundation for the earlier speculations, and to generalize the treatment to include the even-ordered subharmonics.

We shall be concerned only with periodic solutions of (1), and we use the following definitions. (I) If the Fourier expansion of the periodic solution of (1) contains a term $A_r \cos(\omega_r t + \varphi_r)$, where $\omega_r = \omega/r$ and r is an integer, and if $A_r \neq 0$, the solution is said to be *subharmonic* of order $1/r$. (II) If the solution is subharmonic of order $1/r$ it may be written as

$$x = A_r \cos\left(\frac{\omega}{r} t + \varphi_r\right) + \sum_{i \neq r} A_i \cos(\omega_i t + \varphi_i).$$

If in this solution $|A_r| \gg |A_i|$ for all i, the solution is said to be a *strong* subharmonic of order $1/r$. (III) If $A_r \neq 0$ and $A_i = 0$ for all i, the solution is called a *pure* subharmonic. (IV) If $A_r \neq 0$, $A_i = 0$ for all i, and $\varphi_r = 0$ as well, the solution is said to be a *simple* subharmonic.

In view of these definitions, the case of $r = \pm 1$ is not excluded. Therefore, the harmonic response is in the class of subharmonics; in fact, it is the subharmonic of order 1. Moreover, it is quite possible for a solution to contain Fourier components of several distinct subharmonic frequencies. Such solutions will be called *multiply* subharmonic.

Special interest centers on periodic solutions of (1) which are of period $2\pi r/\omega$ where r is an integer. They are subharmonic in the sense of our definition and arise when $A_i = 0$ for all $i < r$, except perhaps when $i = 0$. Such solutions may be written in the form

$$x = B_0 + \sum_{j=1,2,\cdots} B_j \cos\left(\frac{j\omega}{r} t + \varphi_j\right)$$

and they will be called *subharmonic steady states*. They are, in general, multiply subharmonic unless r is a prime number, and $B_0 = 0$. Evidently, the simple and pure subharmonics are subharmonic steady states.

The simple subharmonics. Under the restriction that $f(x)$ is analytic we will show that to every simple subharmonic

$$x = x_0 \cos \frac{\omega}{r} t, \tag{2}$$

where x_0 is a non-vanishing, real, finite but otherwise arbitrary constant, and r is an integer, there belongs one and only one equation

$$x^{\cdot\cdot} + f_r(x) = P_0 \cos \omega t \tag{3}$$

capable of producing it. Furthermore, the "elastic force" in that equation is of the form

$$f_r(x) = \sum_n^r \gamma_r^n x^n \tag{4}$$

and the coefficients γ_r^n, $(n = 1, 2, \cdots, r)$ are uniquely determined by the integer r and by the parameters of the equation. Moreover, any "equation of a simple subharmonic" (i.e., any equation possessing a simple subharmonic as a solution) is reducible to a one-parameter equation.

In view of (2) and a well-known trigonometric identity we have, when $r > 2$ is an integer*,

*Excluding negative integers is no restriction since the cosine is an even function.

$$\cos \omega t = 2^{r-1}\left(\frac{x}{x_0}\right)^r - \frac{r}{1!}2^{r-3}\left(\frac{x}{x_0}\right)^{r-2} + \frac{r(r-3)}{2!}2^{r-5}\left(\frac{x}{x_0}\right)^{r-4}$$
$$- \frac{r(r-4)(r-5)}{3!}2^{r-7}\left(\frac{x}{x_0}\right)^{r-6} + \frac{r(r-5)(r-6)(r-7)}{4!}2^{r-9}\left(\frac{x}{x_0}\right)^{r-8} - \cdots$$

with the series terminating when one of the coefficients vanishes. For $r = 1$ and $r = 2$ we have, respectively, $\cos \omega t = x/x_0$ and $\cos \omega t = 2(x/x_0)^2 - 1$. Since $\ddot{x} = -(\omega/r)^2 x$ we find for the elastic force, when $r > 2$,

$$f_r(x) = \frac{x_0\omega^2}{r^2}\left(\frac{x}{x_0}\right) + P_0\left[2^{r-1}\left(\frac{x}{x_0}\right)^r - \frac{r}{1!}2^{r-3}\left(\frac{x}{x_0}\right)^{r-2} + \frac{r(r-3)}{2!}\left(\frac{x}{x_0}\right)^{r-4}\right.$$
$$\left. - \frac{r(r-4)(r-5)}{3!}2^{r-7}\left(\frac{x}{x_0}\right)^{r-6} + \frac{r(r-5)(r-6)(r-7)}{4!}2^{r-9}\left(\frac{x}{x_0}\right)^{r-8} - \cdots\right]. \tag{5}$$

When $r = 1$, $f_1(x) = (x_0\omega^2 + P_0)\ (x/x_0)$ and when $r = 2$,

$$f_2(x) = \frac{x_0\omega^2}{4}\left(\frac{x}{x_0}\right) + P_0\left[2\left(\frac{x}{x_0}\right)^2 - 1\right].$$

It will be seen that the coefficient of P_0 is in every case a polynomial of degree r containing only even, or only odd, powers according as r is an even or an odd integer. For every positive integer r, the elastic force is also a power series of degree r. When r is odd we have

$$f_r(x) = \frac{x_0\omega^2}{r^2}\left(\frac{x}{x_0}\right) + P_0 \sum_{n=1,3,\cdots}^{r} \alpha_r^n\left(\frac{x}{x_0}\right)^n \tag{6}$$

and when r is even,

$$f_r(x) = \frac{x_0\omega^2}{r^2}\left(\frac{x}{x_0}\right) + P_0 \sum_{n=0,2,4,\cdots}^{r} \alpha_r^n\left(\frac{x}{x_0}\right)^n, \tag{7}$$

where the α_r^n depend on r only and are defined in an obvious manner through (5). It is an interesting fact that $f_r(x)$ is an odd power series when r is odd, but it is not even for even r.

We have shown that the set of equations having simple subharmonic solutions of orders $1/r$ are

$$\ddot{x} + \frac{x_0\omega^2}{r^2} + P_0 \sum_n^r \alpha_r^n\left(\frac{x}{x_0}\right)^n = P_0 \cos \omega t, \qquad (r = 1, 2, \cdots). \tag{8}$$

If we set $\omega t = \tau$, $x/x_0 = \xi$ and $P_0/(x_0\omega^2) = k$, the simple subharmonics become $\xi = \cos \tau/r$, $(r = 1, 2, \cdots)$ and (8) becomes

$$\ddot{\xi} + r^{-2}\xi + k \sum_n^r \alpha_r^n \xi^n = k \cos \tau, \qquad (r = 1, 2, \cdots). \tag{9}$$

These equations contain only the single parameter k. In view of the definition of k and the restrictions on P_0, x_0 and ω it is evident that all finite, non-zero values of k are admitted, but $k = 0$ is excluded. In Table 1 we tabulate the coefficients α_r^n for the equations of the simple subharmonics of orders $1/r$, $(r = 1, 2, \cdots, 9)$.

TABLE 1

r	α_r^0	α_r^1	α_r^2	α_r^3	α_r^4	α_r^5	α_r^6	α_r^7	α_r^8	α_r^9
1	0	1	0	0	0	0	0	0	0	0
2	−1	0	2	0	0	0	0	0	0	0
3	0	−3	0	4	0	0	0	0	0	0
4	1	0	−8	0	8	0	0	0	0	0
5	0	5	0	−20	0	16	0	0	0	0
6	−1	0	18	0	−48	0	32	0	0	0
7	0	−7	0	56	0	−112	0	64	0	0
8	1	0	−32	0	160	0	−288	0	128	0
9	0	9	0	−120	0	432	0	−576	0	256

The α-coefficients in Table 1 may be constructed from the following relations:

$$\alpha_m^n = 0 \quad \text{for all} \quad n > m$$

$$= 0 \qquad \text{when} \quad m + n = 2p - 1, \qquad (p = 1, 2, \cdots)$$

$$\left.\begin{aligned} \alpha_r^0 &= 0 && \text{when} \quad r = 2p - 1 \\ &= -1 && \text{when} \quad r = 4p - 2 \\ &= +1 && \text{when} \quad r = 4p - 4 \end{aligned}\right\} (p = 1, 2, \cdots)$$

$$\alpha_r^1 = r(-)^{(r-1)/2} \text{ when } \quad r = 2p - 1, \qquad (p = 1, 2, \cdots)$$

$$\alpha_p^p = 2\alpha_{p-1}^{p-1}, \qquad (p = 2, 3, \cdots)$$

$$\alpha_m^n = 2\alpha_{m-1}^{n-1} - \alpha_{m-2}^n \quad \text{when} \quad m \geq n + 2$$

$$\text{when} \quad m + n \neq 2p - 1, \qquad (p = 1, 2, \cdots)$$

Finally, we observe that it is evident from the construction of the equation of the simple subharmonic $\xi = \cos r/r_0$ (where r_0 is a positive integer) that, if $f(\xi)$ is analytic, there exists one and only one equation of the form $\xi^{\cdot\cdot} + f(\xi) = k \cos \tau$ capable of producing it; this is the equation $\xi^{\cdot\cdot} + f_{r_0}(\xi) = k \cos \tau$, where

$$f_{r_0}(\xi) = r_0^{-2} + k \sum_n^{r_0} \alpha_{r_0}^n \xi^n$$

and the $\alpha_{r_0}^n$ are defined above.

We would like to discuss briefly the significance of the equations of the simple subharmonics.

If we are asked to find the "steady-state" periodic solutions (if any) of $x^{\cdot\cdot} + \sum_n \alpha^n x^n = P_o \cos \omega t$, we can do this by straightforward, elementary means only if $\sum_n \alpha^n x^n = \alpha^1 x$. For that case we find $x = x_0 \cos \omega t$, where $x_0 = P_0/(\alpha^1 - \omega^2)$, $[(\alpha^1)^{1/2} \neq \pm \omega]$ Now, this last relation is an equation of the sort $g(P_0, \alpha^1, \omega, x_0) =$ const., solved for the amplitude x_0. The method of obtaining this important relation involves techniques which fail unless $\alpha^2 = \alpha^3 = \cdots = 0$. In fact, the failure of these techniques in every but the linear case, presents one of the fundamental difficulties in the discussion of non-linear equations.

There is, however, another obvious way of finding the relation $g(P_0, \alpha^1, \omega, x_0) =$

const., and this way can be taken in the non-linear as well as the linear case. If we are asking for that function $f_1(x)$ for which $x^{\cdot\cdot} + f_1(x) = P_0 \cos \omega t$ has the solution $x = x_0 \cos \omega t$ we find by direct substitution that $f_1(x) = \alpha^1 x$, where $\alpha^1 = (P_0 + \omega^2 x_0)/x_0$; but this last is precisely the equation $g =$ const. solved here for α^1 instead of x_0. If we make the transformation from t to τ, from x to ξ, and put $P_0/(x_0\omega^2) = k$, the desired solution becomes $\xi = \cos \tau$, $f_1(\xi) = \alpha^1\xi$, and $\alpha^1 = 1 + k$. In view of the arbitrariness of x_0, P_0 and ω, this latter result preserves the generality of the former. As a consequence of this method of establishing a correspondence between equations of the form $\xi^{\cdot\cdot} + f(\xi) = k \cos \tau$ (with $f(\xi)$ analytic) and solutions $\xi = \cos \tau/r$ (with r and integer) we find that the *linear*, sinusoidally driven oscillator equation is not the only one having a "steady-state" sinusoidal response; instead, there is an infinite set of such equations. They are the "equations of the simple subharmonics," and the linear equation is simply that of the simple subharmonic of order 1.

It may be disappointing to those who cherish general solutions to equations with arbitrary parameters that the equations of the simple subharmonics have very special coefficients. We share this disappointment. However, a moment of reflection shows that these "special coefficients" actually define the relations which must exist between the parameters of the equations and the amplitude and frequency of the "steady-state" response. In this sense, these equations are no more special than the linear oscillator equation under simple harmonic excitation. Even the technique of prescribing amplitude and frequency of the solution before obtaining additional results is familiar in non-linear vibration problems. For instance, when searching for the harmonic response of the non-linear harmonically driven oscillator equation, the relation which exists between the amplitude of the response (of prescribed frequency!) on the one hand, and the frequency of the excitation on the other is normally obtained by prescribing the amplitude and computing the frequency [20]. Here, we have merely gone one step further and have inserted the governing relations in the equations of motion.

Stability of the simple subharmonics. It is well known that the stability of the periodic solution $\xi = \cos \tau/r$ of (8) depends on the stability of solutions of a Hill equation. In the case of the simple subharmonics, the stability investigation gains from discussing the odd and even subharmonics separately. Accordingly, we begin with an examination of the stability of the simple, *odd* subharmonics.

The stability of the simple subharmonic $\xi = \cos \tau/r$ of the equation

$$\xi^{\cdot\cdot} + r^{-2}\xi + k \sum_{n=1,3,\cdots}^{r} \alpha_r^n \xi^n = k \cos \tau, \qquad (r = 1, 3, \cdots)$$

is identified with the stability of solutions of

$$u^{\cdot\cdot} + \left\{ r^{-2} + k\left[r(-)^{(r-1)/2} + \sum_{n=3,5,\cdots}^{r} n\alpha_r^n \cos^{n-1} \tau/r \right] \right\} u = 0. \tag{10}$$

In this equation, the term belonging to $n = 1$ has been written ahead of the summation, and use has been made of the relation $\alpha_r^1 = r(-)^{(r-1)/2}$ for all odd r.

When $m \geq 2$ is an even integer, we may write

$$\cos^m \frac{\tau}{r} = 2^{-(m-1)} \left\{ \frac{1}{2} \binom{m}{\frac{m}{2}} + \sum_{i=0,1,2,\cdots}^{(m-2)/2} \binom{m}{i} \cos (m - 2i) \frac{\tau}{r} \right\}$$

and when $\bar{m} \geq 3$ is an odd integer we have

$$\cos^{\bar{m}} \frac{\tau}{r} = 2^{-(\bar{m}-1)} \sum_{i=0,1,2,\cdots}^{(\bar{m}-1)/2} \binom{\bar{m}}{i} \cos (\bar{m} - 2i) \frac{\tau}{r},$$

where the conventional notation for the binomial coefficients has been employed. Application of the first of the above relations reduces the stability problem of the odd simple subharmonics to a discussion of the Hill equation

$$u^{\cdot\cdot} + \left(r^{-2} + k\left\{r(-)^{(r-1)/2} + \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-2}} \left[\frac{1}{2}\binom{n-1}{\frac{n-1}{2}} + \sum_{i=0,1,2,\cdots}^{(n-3)/2} \binom{n-1}{i} \cos (n - 1 - 2i) \frac{\tau}{r}\right]\right\}\right)u = 0. \quad (11)$$

A quantitative discussion of the stability of this equation is not possible in view of the complexity of the periodic coefficient. However, there exists evidence [22] that under certain circumstances the stability of the solutions of (11) is not highly sensitive to the precise form of the periodic coefficient. Accordingly, we shall adopt the viewpoint that the coefficient of u is a periodic function which, for purposes of a stability analysis, can be adequately represented by the leading terms of its Fourier expansion. One of the circumstances which must be met is, obviously, that the leading term must have the largest coefficient. If we write the second sum in (11) in an ascending order of Fourier terms we have

$$\sum_{i=0,1,2,\cdots}^{(n-3)/2} \binom{n-1}{i} \cos (n - 1 - 2i) \frac{\tau}{r}$$
$$= \binom{n-1}{\frac{n-3}{2}} \cos \frac{2\tau}{r} + \binom{n-1}{\frac{n-5}{2}} \cos \frac{4\tau}{r} + \cdots + \cos \frac{(n-1)\tau}{r}.$$

In this sum, the magnitude of the coefficients decreases continually—i.e.,

$$\left|\binom{n-1}{i}\right| > \left|\binom{n-1}{i-1}\right|$$

for all integers i in $0 \leq i \leq (n-3)/2$. If only the leading term of the Fourier expansion of

$$\sum_{i=0,1,2,\cdots}^{(n-3)/2} \binom{n-1}{i} \cos (n - 1 - 2i) \frac{\tau}{r}$$

is retained, the stability of the odd simple subharmonics depends on that of the solutions of

$$\frac{d^2u}{dz^2} + (a_0^r + b_0^r \cos z)u = 0,$$

where

$$a_0^r = \frac{1}{4} + \frac{kr^2}{4}\left[r(-)^{(r-1)/2} + \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-1}{2}}\right]$$
$$b_0^r = \frac{kr^2}{2} \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-3}{2}}. \quad (12)$$

Now it can be shown that

$$\left.\begin{aligned} \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-1}{2}} &= r \quad \text{when} \quad r = 4m-1 \\ &= 0 \quad \text{when} \quad r = 4m-3 \end{aligned}\right\} \qquad (m = 1, 2, \cdots)$$

$$\sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-3}{2}} = r.$$

As a consequence, the coefficients in (12) become simply

$$\left.\begin{aligned} a_0^r &= \frac{1}{4}(1 - r^3k) \\ b_0^r &= \frac{1}{2} r^3 k \end{aligned}\right\} \qquad (r = 3, 5, \cdots). \tag{13}$$

The elimination of the parameter k between them yields for all odd integers $r \geq 3$ the relation

$$a_0^r = \frac{1}{4} + \frac{1}{2} b_0^r . \tag{14}$$

We call (14) the "stability characteristic" of the simple odd subharmonics of order $1/r$. It has the remarkable property of being independent of the order $1/r$. This stability characteristic is shown together with the stable and unstable domains of the ab-plane in Fig. 1. From it, we see that simple odd subharmonics have a stable range when

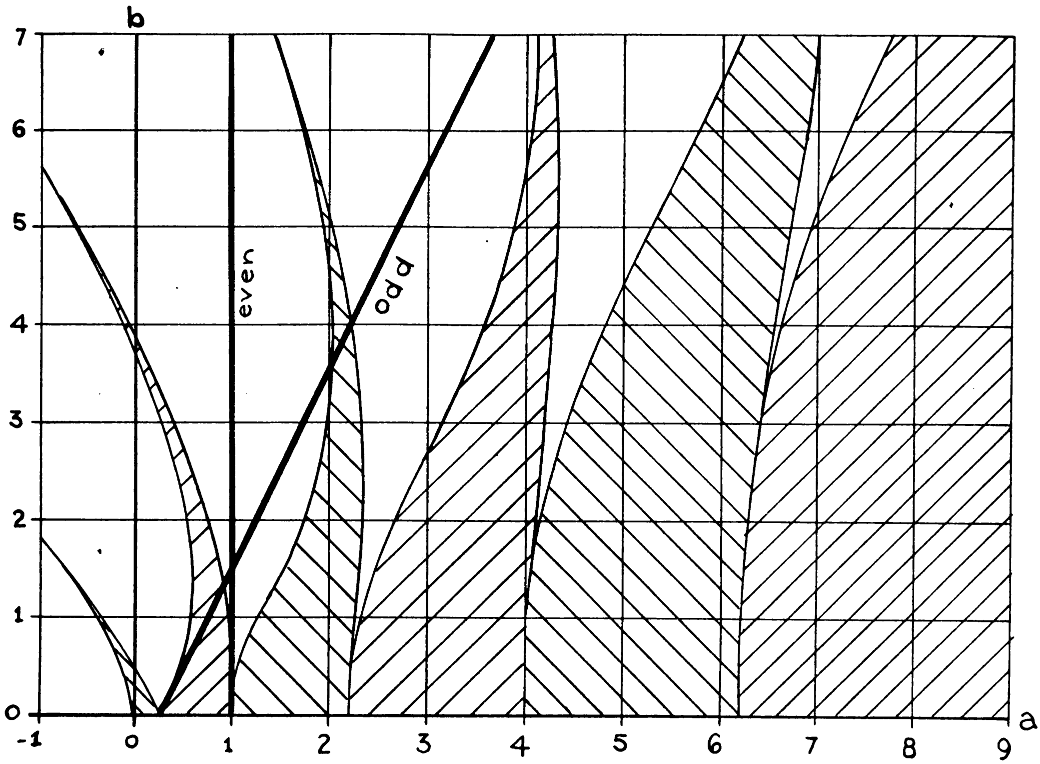

FIG. 1

$1/4 < a_0^r < 1$ and $0 < b_0^r < 3/2$ where the upper bounds are somewhat larger than the least upper bounds. As a_0^r and b_0^r increase, there follows a fairly extensive unstable range for $1 < a_0^r < 2$ and $3/2 < b_0^r < 5/2$, but near $a_0^r = 2$, $b_0^r = 5/2$ a very narrow stable range exists. This is again followed by an extensive unstable range and an extremely narrow stable one near $a_0^r = 4$, $b_0^r = 15/2$.

One of the interesting features of this result is that the simple odd subharmonics are never stable when k is negative. In fact the bounds on k which insure stability (in the first stability range) are $0 < k < 3/r^3$ where, again, the upper bound exceeds the least upper bound somewhat. In the case $r = 3$, for instance, positive k implies a hard spring, negative k a soft spring;* it is seen then that simple subharmonics of order 1/3 can not occur in a system with a soft spring.

Another interesting result is that the simple subharmonic may be stable when the equilibrium position $\xi = 0$ is not. For instance, in the case $r = 3$, the equilibrium position becomes unstable for $k > 1/27$ while the subharmonic is stable until $k = 3/27$, nearly.

We examine, now the stability of the *even* simple subharmonics. When a procedure is followed exactly analogous to the preceeding one, except that r is even, it is found that the stability of the even simple subharmonics is identified with the stability of solution of the equation

$$\frac{d^2u}{dz^2} + (\bar{a}_0^r + \bar{b}_0^r \cos z)u = 0$$

and the coefficients are

$$\bar{a}_0^r = 1, \qquad \bar{b}_0^r = 2r^3k. \tag{15}$$

It is seen that the stability characteristic of the even simple subharmonics is a straight line parallel to the b-axis and crossing the a-axis at $a = 1$. From Fig. 1, where this stability characteristic is shown, we deduce that the even simple subharmonics are almost never stable except in some very narrow ranges the first of which occurs near $|\bar{b}_0^r| = 8$ when $k = \pm 4/r^3$, approximately.

The strong subharmonics. In this section we inquire into the properties of solutions of equations which "lie in the neighborhood" of those of the simple subharmonics. Specifically, we consider the equation

$$\xi^{\cdot\cdot} = g(\tau, \xi, |\mu| \xi^{\cdot}, \mu) = g(\tau + T, \xi, |\mu| \xi^{\cdot}, \mu), \tag{16}$$

with $|\mu|$ small and $g(\tau, \xi, 0, 0) = -f_r(\xi) + k \cos \tau$. The function g is analytic in all variables. When $\mu = 0$, (16) has the solution $\xi = \xi_0(\tau) = \xi_0(\tau + T)$, but T need not be the least period of ξ_0 or g. When μ does not vanish, (16) has the solution $\xi = \eta(\tau, \mu)$. It can, then, be shown [2] that (16) has a solution $\xi = \eta(\tau, \mu)$ which is periodic in τ of period T and analytic in both arguments. Moreover, there is only one such solution for each μ. It should be noted that (16) includes as a special case the equation $\xi^{\cdot\cdot} + |\mu| \xi^{\cdot} + f_r(\xi) = k \cos \tau$, so that it has been shown that small positive damping does not have a profound effect on solutions obtained when damping is neglected.

Since $\eta(\tau, \mu)$ is analytic in the small parameter μ one may write

$$\eta(\tau, \mu) = \eta_0(\tau, 0) + \sum_i \mu^i \eta_i(\tau, 0) = \xi_0(\tau) + \sum_i \mu^i \xi_i(\tau),$$

*In hard springs, the stiffness increases with deflection; in soft springs it decreases with deflection.

where every ξ_i must be periodic since η is periodic. Thus, we have

$$\eta(\tau, \mu) = \cos\frac{\tau}{r} + \sum_i \sum_j \mu^i\left(a_{ij} \cos\frac{j\tau}{r} + b_{ij} \sin\frac{j\tau}{r}\right) \tag{17}$$

and, in view of the earlier definition, $\eta(\tau, \mu)$ is in general a strong subharmonic steady state.

As a special case of an odd strong subharmonic we consider a generalization of Duffing's equation

$$\ddot{\xi} + r^{-2}\xi + k \sum_{n=1,3,\cdots}^{r} \alpha_r^n \xi^n + \mu\xi^{r+2} = k \cos\tau, \qquad (r = 1, 3, \cdots,), \quad |\mu| \text{ small.} \tag{18}$$

It will be noticed that this reduces to Duffing's equation when $r = 1$.

When the theory of (17) is applied to (18), the solution to the latter, within linear terms in μ, turns out to be

$$\xi = \left[1 - \mu\binom{r+2}{\frac{r+1}{2}}\Big/ 2^{r+2}kr\right] \cos\frac{\tau}{r} + \mu \sum_{j=3,5,\cdots}^{r+2} a_{1j} \cos\frac{j\tau}{r}, \tag{19}$$

where for $j = 3, 5, \cdots, \leq (n-1)/2$

$$a_{1i} = -\binom{r+2}{\frac{r+2-j}{2}}\Big/ \left\{2^{r+1}\left[(1-j^2)r^{-2} + k\left(r + \sum_{n=3,5,\cdots}^{r} \frac{n\alpha_r^n}{2^{n-1}}\binom{n-1}{\frac{n-1}{2} - j}\right)\right]\right\}$$

and for $j > (n-1)/2, \cdots, r-2, r, r+2$

$$a_{1j} = -\binom{r+2}{\frac{r+2-j}{2}}\Big/ \{2^{r+1}[(1-j^2)r^{-2} + kr]\},$$

where the usual notation for the binomial coefficients has been used. The solution (19) is in general multiply subharmonic. Suppose $r = r_0$ to be odd integer which is not a prime. Let it have factors $j_0, j_1, \cdots, j_m$ such that the ratios $r_0/j_{01}, \cdots, r_0/j_m$ are integers $c_0, \cdots, c_m$. Then (19) becomes

$$\xi = \left[1 - \mu\binom{r_0+2}{\frac{r_0+1}{2}}\Big/ 2^{r_0+2}kr_0\right] \cos\frac{\tau}{r_0} + \mu \sum_{i=0}^{m} a_{1j_i} \cos\frac{\tau}{c_i} + \mu \sum_{j \neq j_i} a_{1j} \cos\frac{j\tau}{r_0}, \tag{20}$$

where the first term is strongly subharmonic of order $1/r_0$, the terms of the first sum are subharmonic of orders $1/c_i$, $(i = 0, 1, \cdots, m)$ and the terms in the second sum are not subharmonic. This explains why one can find subharmonics of many orders in the solution of a non-linear oscillator equation.

The stability of the solution (19) depends on that of a Hill equation. When only the leading terms of the periodic coefficient are retained, the "stability equation" becomes

$$\frac{d^2u}{dz^2} + [(a_0^r + \mu a_1^r) + (b_0^r + \mu b_1^r) \cos z]u = 0,$$

where a_0^r and b_0^r are given in (13), and

$$a_1^r = \left[r^2 \binom{r+2}{\frac{r+1}{2}} \Big/ 2^{r+2} \right] \{(n_0 + 1)[1 - (-)^{(r-1)/2}]/4 + (r + 2)\},$$

$$b_1^r = \left[r^2 \binom{r+2}{\frac{r+1}{2}} \Big/ 2^{r+2} \right] \left\{ (n_1 - 4)/4 + \binom{r+2}{\frac{r-1}{2}} (r + 2) \right\},$$

$$n_0 = \sum_{n=3,5,\cdots}^{r} \frac{n^2 \alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-1}{2}} \Big/ \{r[1 - (-)^{(r-1)/2}]\},$$

$$n_1 = \sum_{n=3,5,\cdots}^{r} \frac{n^2 \alpha_r^n}{2^{n-1}} \binom{n-1}{\frac{n-3}{2}} \Big/ r.$$

It can be shown easily that n_0 and n_1 as well as a_1^r and b_1^r are positive for every r. The relation between a_1^r and b_1^r becomes a little more transparent when one writes

$$\binom{r+2}{\frac{r+1}{2}} = \binom{r+2}{\frac{r-1}{2}} \kappa(r)$$

in the expression for b_1^r . Since

$$\binom{r+2}{\frac{r+1}{2}} \geq \binom{r+2}{\frac{r-1}{2}}$$

the quantity $\kappa(r) \geq 1$. In fact, $\kappa(3) = 2$ and $\kappa(r) \to 1$ monotonically as $r \to \infty$. We now have

$$b_1^r = \left[r^2 \binom{r+2}{\frac{r+1}{2}} \Big/ 2^{r+2} \right] \{(n_1 - 4)/4 + \kappa(r)^{-1}(r + 2)\}$$

from which it is seen by comparing it with the expression for a_1^r that $a_1^r/b_1^r \to 1$ as r increases. For the likely conjecture that, $n_0 = n_1 = r/2$ approximately, the relation between a_1^r and b_1^r may be expected to be as shown in Fig. 2. It follows, that the stability characteristic of (18) lies everywhere in the neighborhood of the "odd stability characteristic" of Fig. 1 but has a slightly different intercept with the a-axis. As a consequence, *stable* strong subharmonic steady states can occur for negative k. In particular, when $r = 3$, *strong* subharmonics may be stable in systems with soft springs while *simple* subharmonics are not, as shown earlier.

When the case of even-ordered subharmonics, analogous to the generalized Duffing equation is examined the equation may be written as

$$\xi^{\cdot\cdot} + r^{-2}\xi + k(-)^{r/2} + k \sum_{n=2,4,\cdots}^{r} \alpha_r^n \xi^n + \mu \xi^{r+2} = k \cos \tau, \qquad (r = 2, 4, \cdots) \qquad (21)$$

and the solution within linear terms in μ becomes

$$\xi = -\mu \frac{r^2}{2^{r+1}} \binom{r+2}{\frac{r+2}{2}} + \cos \frac{\tau}{r} + \mu \sum_{j=2,4,\cdots}^{r+2} a_{1j} \cos \frac{j\tau}{r}, \tag{22}$$

where

$$a_{1j} = -\binom{r+2}{\frac{r+2-j}{2}} \Big/ \{2^{r+1}[(1-j^2)r^{-2}]\}.$$

The stability of this solution depends (within the earlier simplification) on the solutions of

$$\frac{d^2u}{dz^2} + [(\bar{a}_0^r + \mu\bar{a}_1^r) + (\bar{b}_0^r + \mu\bar{b}_1^r) \cos z]u = 0,$$

where $\bar{a}_0^r$ and $\bar{b}_0^r$ are given in (15) and

$$\bar{a}_1^r = r^2 a_{10}\left[k \sum_{n=2,4,\cdots}^{r} \frac{n(n-1)\alpha_r^n}{2^{n-2}} \binom{n-2}{\frac{n-2}{2}} + \frac{r+2}{2^r}\binom{r}{r/2}\right],$$

$$\bar{b}_1^r = r^2 a_{10} \frac{r+2}{2^r}\binom{r+1}{r/2},$$

$$a_{10} = (r^2/2^{r+1})\binom{r+2}{\frac{r+2}{2}}.$$

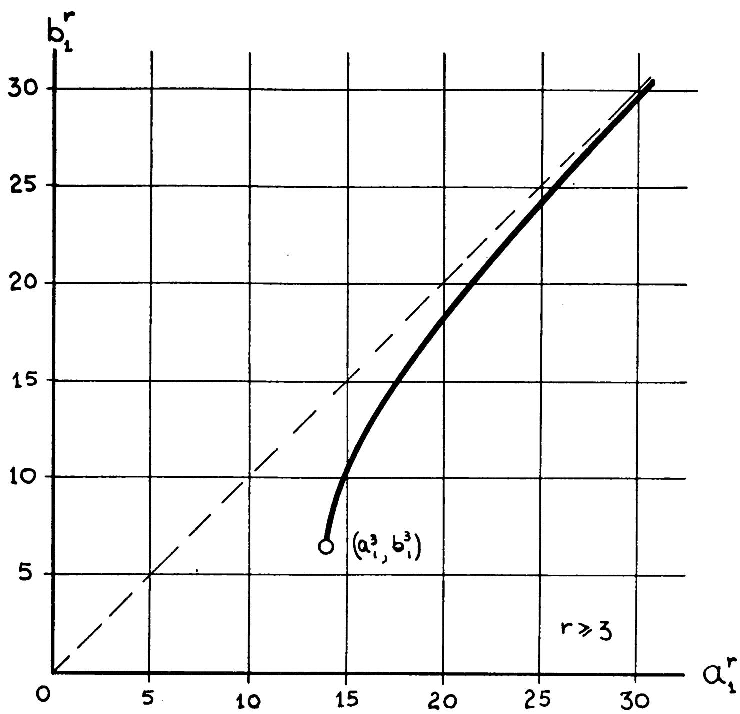

FIG. 2

The effect of these quantities on the stability can be established from the observation that $\bar{a}_1^r \neq 0$. Since $|\mu|$ is small, the stability characteristic will lie near the "even stability characteristic" of Fig. 1. However, it crosses the a-axis at $a = 1 + \mu\bar{a}_1^r$. Whether $\mu\bar{a}_1^r$ is positive or negative, the characteristic will pass through a stable range of the ab-plane for some small values of k in the interval $|k| > 0$. Thus, while *simple* even subharmonics are almost never stable, *strong* even subharmonics are easily stable.

Concluding remarks. We have concerned ourselves with the stable (and unstable) state of systems whose equations are special cases of

$$\xi^{\cdot\cdot} = F(\xi, \xi^{\cdot}, \mu, \tau) = F(\xi, \xi^{\cdot}, \mu, \tau + T),$$

where F is analytic in all variables on any interval, but in general not linear in them. Trefftz has remarked [21] that stable solutions tend with $\tau \to \infty$ to periodic solutions of period rT where r is an integer. In this paper, these solutions are called subharmonic for every r.

In spite of significant progress toward a better understanding of this equation in its general form [6, 7, 9, 11], or when it is nearly linear [16, 18], one cannot in general predict which among a variety of possible periods the solution will exhibit. Nor is the "mechanism" of subharmonic vibrations understood. With the mechanism we mean what Stoker [20] calls "a plausible physical explanation." An understanding of the mechanism seems to us essential for a profound and satisfying understanding of the phenomenon.

The latter difficulty has been removed—at least in the case of the simple subharmonic steady states—by the formulation of the equations of the simple subharmonics. Evidently, if we are ready to believe that the equation of the simple subharmonic of order 1 is the only one capable of producing the simple harmonic response, we should be ready to accept a similar statement for the simple subharmonics of any order.

Moreover, the formulation of these equations has robbed the linear equation of the singular position which it is usually thought to occupy as the only one whose solution is purely sinusoidal, or as the only one whose steady state has a uniquely determined period. For, it appears that the equations of the simple subharmonics of any order $1/r$ have a steady state $\xi = \cos \tau/r$ of unique frequency $1/r$. (We stipulate here that the steady state remains unchanged when the argument τ/r is replaced by $\tau/r + 2\pi n$, $(n = 1, 2, \cdots, r - 1)$; i.e., when the phase of the response is shifted by n cycles of the excitation.) This result is deduced as follows: the conventional method of determining the steady states (exclusive of their stability characteristics) is to assume as the solution a periodic function in τ in the form of a Fourier series with undetermined coefficients. The coefficients are, then, fixed by the relations which are obtained when the solution is substituted in the equation [20]. If this technique is used on the equation of the simple subharmonic of order $1/r$, and the period T of the assumed solution is commensurate with, but larger than $2\pi r$,—i.e., $T/2\pi r > 1$ is a rational number—it will evidently turn out that all coefficients vanish except that of $\cos \tau/r$ whose coefficient is 1. When the period T is not commensurate with $2\pi r$ the solution is not periodic in τ since the coefficient of $\cos \tau/r$ does not vanish. It is, in fact, almost periodic. This result bears out the conjecture [3, 13], that subharmonics of order $1/r$ can occur in equations of the form $\xi^{\cdot\cdot} + f(\xi) = k \cos \tau$ only if $f(\xi)$ is a power series of degree r or higher.

Finally, the multiplicity of steady states in non-linear equations of the type under discussion appears to have lost some of its mystery. An equation of the form $\xi^{\cdot\cdot} + f(\xi) =$

$k \cos \tau$ either is, or it is not, that of a simple subharmonic of order $1/r_0$. If it is, the steady state appears to be unique. If it is not, the steady states are $1/r_0$, $1/r_1$, $\cdots$, where the r_i are integers and $r_i > r_{i-1}$. However, the steady states of orders $1/r_1$, $1/r_2$, $\cdots$, are now merely the *higher order terms of the Fourier development* of a solution of period $2\pi r_0$. This observation by itself is, however, not sufficient to explain the occurrence of a multiplicity of subharmonics. For instance, a solution $\sum_i A_i \cos i\tau/r$ is periodic of order $1/m > 1/r$ only if m is a factor of r and, if the $A_i = 0$ for all $i = 1, 2, \cdots, (r/m) - 1$. Now, it is obvious that distinct periodic solutions of one and the same equation have different initial conditions (since the equations considered here satisfy the conditions for existence and uniqueness of solutions when the initial conditions are prescribed.) Therefore, one can explain the existence of distinct periodic solutions by regarding the Fourier coefficients in the solution $\sum_i A_i \cos i\tau/r$ as functions of the initial conditions. Let the initial conditions of the $1/m$th subharmonic be denoted by $\xi_m(0)$, $\dot{\xi}_m(0)$ where m is a factor of r. Then the subharmonic of order $1/m$ will be observed if it is stable, and if

$$A_1(\dot{\xi}_m(0), \dot{\xi}_m(0)) = \cdots = A_{(r/m)-1}(\dot{\xi}_m(0), \dot{\xi}_m(0)) = 0 \quad \text{while} \quad A_{r/m}(\xi_m(0), \dot{\xi}_m(0) \neq 0.$$

This observation does not aid in the construction of distinct subharmonic steady states; however, it provides a satisfactory mechanism for explaining them.

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