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CROCCO'S VORTICITY LAW IN A NON-UNIFORM MATERIAL*

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In the derivation of Crocco's vorticity law [1] there is a basic restriction, that the material in the flow field be uniform, that the flow be isocompositional. This restriction is not generally appreciated, and is only implied in most published derivations of this law. The purpose of this note is to express the corresponding form of the vorticity law valid with non-uniform material, such as a gas mixture with non-zero concentration gradients or a material in which a chemical reaction is proceeding.

The basic assumptions which are needed are: (i) no body forces except gravity; (ii) no frictional stresses; (iii) steady flow; (iv) thermodynamic quasi-equilibrium; and (v) continuous first derivatives of thermodynamic variables and velocity. With assumption (iv) the material obeys a differential state equation of the form

$$dh = T ds + \frac{dp}{\rho} + \sum \mu_k dn_k, \quad (1)$$

where μ_k and n_k are the molar chemical potential and the specific molar concentration, respectively, of the k th constituent, p and T are the pressure and absolute temperature, and h , s , and $1/\rho$ are the specific enthalpy, entropy and volume.

Since the material everywhere in the flow field obeys Eq. (1), we may write the corresponding gradient equation

$$\nabla h = T \nabla s + \frac{\nabla p}{\rho} + \sum \mu_k \nabla n_k. \quad (2)$$

With assumptions (i), (ii) and (iii) we may write the equation of motion of the fluid

$$\nabla \frac{q^2}{2} - \mathbf{q} \times (\nabla \times \mathbf{q}) + \frac{\nabla p}{\rho} + \nabla \Omega = 0, \quad (3)$$

where Ω is the gravitational potential. The total enthalpy h_0 is defined

$$h_0 = h + \frac{q^2}{2} + \Omega. \quad (4)$$

Eqs. (2), (3), and (4) may be combined to give the desired vorticity law

$$\mathbf{q} \times (\nabla \times \mathbf{q}) = \nabla h_0 - T \nabla s - \sum \mu_k \nabla n_k. \quad (5)$$

If there is diffusion present the velocity is defined as a mass mean, as the total mass flow vector divided by the total density.

In the absence of any real-fluid effects in an isocompositional flow, both h_0 and s are constant along streamlines, and ∇h_0 and ∇s are both directed normal to \mathbf{q} . These real-fluid effects are viscosity, excluded by assumption (ii), heat conduction, and relaxation.

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With a non-uniform material $\sum \mu_k dn_k$ is zero along a streamline if there is no diffusion and if any chemical reaction is reversible. Thus, since diffusion and irreversibility of chemical reaction are real-fluid effects associated with varying composition, no new feature appears in the generalized form of Crocco's vorticity law.

The vorticity law is most useful if h_0 and the n_k 's are constant over the entire flow field; in this case the entropy gradient may be obtained directly from the vorticity or the lateral components of the vorticity from the entropy gradient. However, with no real-fluid effects present and with sufficient knowledge of the flow field the vorticity law still gives a useful relation between vorticity and entropy gradient, as then h_0 is a function only of the streamline and the n_k 's may in principle be obtained, and the lateral gradients of these quantities may be calculated.

In a binary mixture we may set

$$c = n_1 M_1, \quad (6a)$$

$$1 - c = n_2 M_2, \quad (6b)$$

$$\mu = \frac{\mu_1}{M_1} - \frac{\mu_2}{M_2}, \quad (6c)$$

where M_1 and M_2 are the molecular weights (molar mass) of the two components. With this notation we have

$$\sum \mu_k dn_k = \mu dc \quad (7)$$

and Crocco's vorticity law takes the form

$$\mathbf{q} \times (\nabla \times \mathbf{q}) = \nabla h_0 - T \nabla s - \mu \nabla c. \quad (8)$$

REFERENCE

- [1] L. Crocco, *Ein neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation*, ZAMM 17, 1-7 (1937)

THE SCHWARZIAN DERIVATIVE AND THE APPROXIMATION METHOD OF BRILLOUIN*

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1. Let the coefficient function of the differential equation

$$x'' + f^2(t)x = 0 \quad (1)$$

be the square of a positive function $f(t)$ which is given, and possesses a continuous second derivative, for large positive t , say for $t_0 \leq t < \infty$. Then the function

$$\varphi(t) = \int_{t_0}^t f(s) ds \quad (2)$$

is strictly increasing and has a continuous third derivative, whereas the function

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