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EXTENSION OF MICHELL'S THEOREM TO PROBLEMS OF PLASTICITY AND CREEP*

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A well known theorem of linear, isotropic elasticity due to J. H. Michell [1] gives the conditions under which the generalized plane stress distribution in a multiply-connected sheet subjected to prescribed boundary stresses is independent of Poisson's ratio. Such independence exists if, and only if, the resultant force (not necessarily the couple) on each boundary vanishes. It is shown in this paper that the same independence of the elastic value of Poisson's ratio holds under the same conditions even when plastic flow and creep occur[1].

Let $\sigma_{\alpha\beta}(x, y; t)$, $\epsilon_{\alpha\beta}(x, y; t)$, and $u_\alpha(x, y; t)$ be the time dependent stress, strain, and single-valued displacement distributions[2] that constitute a solution to the following time-dependent boundary value problem for the region R bounded externally by the curve C_0 and internally by the curves C_i $(i = 1, 2, \cdots N)$. In R:

$$\sigma_{\alpha\beta,\alpha} = 0, \tag{1}$$

$$\epsilon_{\alpha\beta} = \frac{1}{E}\left[(1 + \nu)\sigma_{\alpha\beta} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right] + f_{\alpha\beta}\,, \tag{2}$$

$$\epsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \tag{3}$$

On C_i :

$$\sigma_{\alpha\beta}n_\alpha^{(i)} = T_\beta^{(i)} \qquad (i = 0, 1, 2, \cdots N). \tag{4}$$

In this formulation E is Young's modulus, ν is Poisson's ratio, $n_\alpha^{(i)}$ is the unit outward normal to the boundary C_i , and the $T_\beta^{(i)}$ are prescribed, time-dependent distributions of boundary traction[3]. The functions $f_{\alpha\beta}$ $(= f_{\beta\alpha})$ represent the plasticity and creep contributions to the strain, and are permitted to depend on time and the detailed history of stress. The stress-strain relation (2) therefore represents the most general (isothermal) plasticity and creep law for elastically isotropic materials.

Next, consider a different material obeying the same stress-strain law (2) with the exception that Poisson's ratio is replaced by $\nu^* \neq \nu$; the value of E, and the functional dependence of $f_{\alpha\beta}$ on time and stress history are assumed to remain unchanged. To determine whether or not $\sigma_{\alpha\beta}$ remains a solution of the same boundary value problem for the new material, it is only necessary to see whether the strains

$$\epsilon_{\alpha\beta}^* = \frac{1}{E}\left[(1 + \nu^*)\sigma_{\alpha\beta} - \nu^*\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right] + f_{\alpha\beta} \tag{5}$$

are derivable by means of the strain-displacement equation (3) from some single-valued displacement u_α^* .

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[1]In the case of elasticity, independence of Poisson's ratio is equivalent (by dimensional analysis) to independence of all elastic constants; this is no longer true in the present case of plasticity and creep.

[2]The usual summation convention and subscript notation for tensors and vectors are used here, with Greek subscripts taking on the values 1, 2.

[3]Whether Michell's conditions are fulfilled or not, the $T_\beta^{(i)}$ must, of course, satisfy conditions of equilibrium of the body as a whole.

Now let

$$\epsilon'_{\alpha\beta} = \epsilon_{\alpha\beta} - \epsilon^*_{\alpha\beta} . \tag{6}$$

Then, from (2) and (5),

$$\epsilon'_{\alpha\beta} = \frac{\nu - \nu^*}{E} (\sigma_{\alpha\beta} - \sigma_{\gamma\gamma}\delta_{\alpha\beta}). \tag{7}$$

Next consider the integral

$$J = \frac{E}{\nu - \nu^*} \iint_R \epsilon'_{\alpha\beta}\tau_{\alpha\beta}\, dA, \tag{8}$$

where $\tau_{\alpha\beta}$ is any distribution of stress satisfying equilibrium:

$$\tau_{\alpha\beta,\alpha} = 0 \tag{9}$$

and producing zero boundary tractions:

$$\tau_{\alpha\beta}n_\alpha^{(i)} = 0 \quad \text{on} \quad C_i \qquad (i = 0, 1, 2, \cdots N). \tag{10}$$

Now, if $\epsilon'_{\alpha\beta}$ is derivable from a single-valued displacement, it follows from the principle of virtual work that the integral J given by (8) vanishes. But a converse theorem is also valid; if $J = 0$ for *all* $\tau_{\alpha\beta}$ satisfying (9) and (10), then $\epsilon'_{\alpha\beta}$ is derivable from a single-valued displacement [2, 3][4]. Substituting (7) into (8) gives

$$J = \iint_R (\sigma_{\alpha\beta}\tau_{\alpha\beta} - \sigma_{\alpha\alpha}\tau_{\beta\beta})\, dA. \tag{11}$$

The stresses $\tau_{\alpha\beta}$ can be related to a stress function ϕ by

$$\tau_{\alpha\beta} = \phi_{,\gamma\gamma}\delta_{\alpha\beta} - \phi_{,\alpha\beta} \tag{12}$$

and then (11) becomes

$$J = -\iint_R \sigma_{\alpha\beta}\phi_{,\alpha\beta}\, dA. \tag{13}$$

As a consequence of the boundary conditions (10) on $\tau_{\alpha\beta}$, it follows that $\phi_{,\alpha}$ is single valued, and, furthermore, has a constant value, say $K_\alpha^{(i)}$, on each boundary (see, for example, [5], p. 191). It is therefore possible to transform (13) as follows:

$$\begin{aligned} J &= -\iint_R [(\sigma_{\alpha\beta}\phi_{,\alpha})_{,\beta} - (\sigma_{\alpha\beta,\beta})\phi_{,\alpha}]\, dA, \\ &= -\sum_{i=0}^{N} K_\alpha^{(i)} \oint_{C_i} \sigma_{\alpha\beta}n_\beta^{(i)}\, dS, \\ &= -\sum_{i=0}^{N} K_\alpha^{(i)}P_\alpha^{(i)}\, dS, \quad \text{where} \quad P_\alpha^{(i)} = \oint_{C_i} T_\alpha^{(i)}\, dS. \end{aligned}$$

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[4]The proof in [2] is specifically limited to simply-connected regions; the proof in [3], and the supporting theorems contained in [4], are valid for multiply-connected regions.

Hence, if $P_\alpha^{(i)} = 0$ on each boundary, J vanishes for all admissible choices of $\tau_{\alpha\beta}$. Hence, by the converse theorem mentioned above, $\epsilon'_{\alpha\beta}$ is derivable from a single valued displacement. The same is then necessarily true of $\epsilon^*_{\alpha\beta} = \epsilon_{\alpha\beta} + \epsilon'_{\alpha\beta}$, and hence $\sigma_{\alpha\beta}$ remains a solution for the stress when Poisson's ratio is changed. On the other hand, if $P_\alpha^{(i)}$ does not vanish on some boundaries, a suitable choice of $\tau_{\alpha\beta}$ can always be made to render J non-zero. But this would necessarily imply that the strains $\epsilon'_{\alpha\beta}$ (and hence $\epsilon^*_{\alpha\beta}$) are *not* derivable from a single valued displacement, whence $\sigma_{\alpha\beta}$ would certainly not constitute a solution for the new material.

The present theorem can be useful in simplifying the initial formulation of some problems. For example, the choice $\nu = \frac{1}{2}$ in conjunction with a total stress-strain law of plasticity permits the use of a single formula for the sum of the elastic and plastic components of strain; in other problems, the choice $\nu = 0$ might be more appropriate. In addition, the present theorem may conceivably have significance in connection with photoplasticity.

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ON ISOPERIMETRIC INEQUALITIES IN PLASTICITY*

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Abstract. The purpose of this paper is the proof of the inequality $P \geq 6\pi M_0$, where P is the total limit load, M_0 the yield moment of a thin, perfectly plastic, simply supported, uniformly loaded plate of arbitrary shape and connection.

Introduction. The theory of thin, rigid-perfectly plastic plates, given by Hopkins and Prager [1]** has been applied to circular plates with various load and edge conditions. However, if one tries to extend this theory to non-symmetrical cases, serious difficulties arise in seeking examples of exact solutions, although some cases have been solved (see for instance [2]). As a contribution to the estimation of the limit load in an arbitrary plate we shall use here the isoperimetric inequality, which relates a circular domain to an arbitrary domain in a convenient manner. One of the principal theorems of limit analysis [3] and the methods for isoperimetric problems given in Polya's and Szegö's book [4] will be used. Similar problems have been proposed and solved for other physical quantities, as for example the torsional rigidity, the principal frequency, etc.

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**Numbers in square brackets refer to the bibliography at the end of the paper.