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ON SOME ITERATIVE METHODS FOR SOLVING ELLIPTIC DIFFERENCE EQUATIONS*

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1. Introduction. The numerical solution of boundary value problems for partial differential equations usually requires the solution of large systems of linear equations. The order, n, of such systems is essentially equal to the number of mesh points in the domain under consideration. Since direct inversion procedures require the order of n^3 operations they are not practicable, even using high speed digital computers, for reasonable meshes in two or more dimensions. Thus iterative methods for solving linear systems are of great interest as they usually require the order of n^2 operations. In addition the coefficient matrix of the system which results from the finite difference approximations has many strategically placed zeroes. However, no special account of these zeroes is taken in most direct inversions or in general iterative procedures. It is reasonable to expect that particular methods, designed in accordance with the general structure of the coefficient matrix, could further reduce the number of operations. Many such special iteration schemes have been devised and conditions on the coefficient matrix which are sufficient to insure the convergence of some of these methods have been obtained [1, 7, 9]. However there is no general comparison procedure to determine which of many possible methods is "best" in a given case.

In the present paper we formulate a family of iterative schemes for a particular class of coefficient matrices (in which the zeroes are placed as in the usual five-point Laplace difference equations). This family is defined by a generalization of the usual notion of extrapolation or over-relaxation. It is then possible to formulate the problem of finding the "best" scheme and, more important, some general theorems on the eigenvalues of these schemes are proved.

The theorems are used to define three subclasses, called complete image classes, of the general family. These classes contain many of the schemes in current use as well as generalizations of them. Thus it is shown that a variety of independently proposed and seemingly unrelated iterative methods are special cases of a general class of methods. These complete image classes are such that each of the eigenvalues of any scheme in the class is a given function of one of the eigenvalues of a particular reference scheme of the class. Thus a knowledge of the eigenvalues of the reference scheme permits, in principle, the determination of the best scheme of the given class.

A special class of equations is considered for which all the eigenvalues of each reference scheme can be explicitly written in terms of the eigenvalues of two matrices. It is then

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possible to determine which of the reference schemes is best and hence it is also possible to determine that scheme which is best of all those in the complete image classes.

As an example of the application of this general theory the Laplace difference equations are considered. The well-known results on rate of convergence for the Richardson, Liebmann and extrapolated Liebmann methods are immediate consequences. Analogous results are obtained for the less well-known "line" methods* which are shown to be superior.

It is clear that the methods of the present paper can be applied to more general iterative schemes than those considered. In addition many of the present results can be easily extended to problems in higher dimensions and with more general boundaries.

2. Formulation. A large class of two dimensional linear elliptic difference equations are of the form:

$$\phi_{ij} - l_{ij}\phi_{i-1,j} - r_{ij}\phi_{i+1,j} - b_{ij}\phi_{i,j-1} - t_{ij}\phi_{i,j+1} = s_{ij} \; ; \tag{2.0}$$

within a coordinate rectangle specified by $1 < i < p$, $1 < j < q$. On the boundaries of the domain equations of the form (2.0) hold with the coefficients:

$$l_{1j} = r_{pj} = 0, \qquad 1 \leq j \leq q; \qquad b_{i1} = t_{iq} = 0, \qquad 1 \leq i \leq p. \tag{2.1}$$

Such equations are obtained from second order† elliptic partial differential equations by applying the usual second order difference approximations on some coordinate mesh, $(\xi_i \, , \, \eta_j)$. The coefficient matrix of the resulting system, (2.0) and (2.1) must be non-singular and, with a little care in differencing [1], can be made positive definite (and symmetric if the equation is self adjoint). However, we shall assume, unless otherwise stated, only the non-singularity.

For convenience of notation and discussion we introduce, for each j, the p-dimensional column vectors:

$$\Phi_j \equiv \begin{bmatrix} \phi_{1j} \\ \phi_{2j} \\ \vdots \\ \phi_{pj} \end{bmatrix}, \qquad S_j \equiv \begin{bmatrix} s_{1j} \\ s_{2j} \\ \vdots \\ s_{pj} \end{bmatrix}; \tag{2.2}$$

and the $p \times p$ order matrices:

$$L_j \equiv \begin{bmatrix} 0 & & & & & 0 \\ l_{2j} & 0 & & & & \\ & l_{3j} & 0 & & & \\ & & \cdot & 0 & & \\ & & & \cdot & 0 & \\ & & & & \cdot & 0 \\ 0 & & & & & l_{pj} \; 0 \end{bmatrix}, \quad R_j \equiv \begin{bmatrix} 0 & r_{1j} & & & & 0 \\ & 0 & r_{2j} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & r_{p-1,j} \\ 0 & & & & & 0 \end{bmatrix}, \quad 1 \leq j \leq q; \tag{2.3}$$

*The origin of iterative line methods is obscure. They have been in use by Russian mathematicians for a number of years. About 1945 J. von Neuman and L. H. Thomas independently proposed line methods for parabolic difference equations. An independent investigation was initiated by M. E. Rose and the present author in 1953 and some of the results of Sec. 8 were then obtained. Peaceman and Ratchford have studied their applicability to parabolic difference equations and double-sweep iterative solutions of the Laplace difference equations.

†The five point scheme implied by (2.0) assumes no mixed partial derivates in the equations.

$$B_j \equiv \begin{bmatrix} b_{1j} & & 0 \\ & b_{2j} & \\ & \ddots & \\ 0 & & b_{pj} \end{bmatrix}, \quad 1 < j \leq q; \qquad T_j \equiv \begin{bmatrix} t_{1j} & & 0 \\ & t_{2j} & \\ & \ddots & \\ 0 & & t_{pj} \end{bmatrix}, \quad 1 \leq j < q.$$

The system (2.0–.1) can now be written as

$$\begin{aligned} (I - L_1 - R_1)\Phi_1 - T_1\Phi_2 &= S_1 ; \\ -B_j\Phi_{j-1} + (I - L_j - R_j)\Phi_j - T_j\Phi_{j+1} &= S_j , \qquad 2 \leq j \leq q-1; \\ -B_q\Phi_{q-1} + (I - L_q - R_q)\Phi_q &= S_q . \end{aligned} \tag{2.4}$$

Here we have introduced the identity matrix, I, which is always assumed to be of the same order as the square matrices to which it is added.

Further simplification is obtained by introducing the $(p \times q)$-dimensional column vectors* (or q-dimensional compound vectors):

$$\Phi \equiv \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_q \end{bmatrix} \equiv \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{pq} \end{bmatrix}, \qquad S \equiv \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_q \end{bmatrix} \equiv \begin{bmatrix} s_{11} \\ s_{21} \\ \vdots \\ s_{pq} \end{bmatrix}; \tag{2.5}$$

and the $[(pq) \times (pq)]$-order matrices (or $[q \times q]$-order compound matrices):

$$L \equiv \begin{bmatrix} L_1 & & 0 \\ & L_2 & \\ & \ddots & \\ 0 & & L_q \end{bmatrix}, \qquad R \equiv \begin{bmatrix} R_1 & & 0 \\ & R_2 & \\ & \ddots & \\ 0 & & R_q \end{bmatrix},$$

$$B \equiv \begin{bmatrix} 0 & & & & 0 \\ B_2 & 0 & & & \\ & B_3 & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & \ddots \\ 0 & & & & B_q \; 0 \end{bmatrix}, \qquad T \equiv \begin{bmatrix} 0 & T_1 & & & & 0 \\ & 0 & T_2 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & T_q \\ 0 & & & & & 0 \end{bmatrix}. \tag{2.6}$$

The system of linear equations (2.0–.1), or (2.4) now becomes

$$M\Phi \equiv (I - L - R - B - T)\Phi = S. \tag{2.7}$$

Similar formulations may be introduced for more general boundaries and in higher dimensions. In particular, if the boundaries are composed of coordinate segments, the

*The vector Φ of (2.5) determines an "ordering" [1] of the unknowns, Φ_{ij}; but this order need bear no relationship to the sequence in which the iterative computations are carried out.

matrices L_i and R_i remain square but of different orders while the matrices B_i and T_i become rectangular. Another pair of "neighbors", $\phi_{ijk\pm 1}$, appear in (2.0) with each unit increase of the dimension and, correspondingly, additional pairs of matrices must be introduced in a manner similar to those of (2.6).

3. General single sweep iterations. We summarize here some terminology and known results for a class of iterative methods for solving (2.7); this class is defined as follows: Let the coefficient matrix, M, be written as

$$M = N - P, \tag{3.0}$$

where $| N | \neq 0$. We call this a "splitting" of the coefficient matrix and the system (2.7) becomes

$$N\Phi = P\Phi + S, \tag{3.1}$$

with the formal solution

$$\Phi = (N - P)^{-1}S = (I - N^{-1}P)^{-1}N^{-1}S. \tag{3.2}$$

The iterative procedure is defined, starting from some arbitrary guess, $\Phi^{(0)}$, at the solution vector, by the recursion

$$N\Phi^{(\nu)} = P\Phi^{(\nu-1)} + S. \tag{3.3}$$

Thus in one sweep through the mesh a new iterate is obtained. Applying (3.3) recursively yields for the νth iterate

$$\Phi^{(\nu)} = [I + (N^{-1}P) + (N^{-1}P)^2 + \cdots + (N^{-1}P)^{\nu-1}]N^{-1}S + (N^{-1}P)^{\nu}\Phi^{(0)}. \tag{3.4}$$

Thus [2] $\Phi^{(\nu)} \to \Phi$ as $\nu \to \infty$, for arbitrary $\Phi^{(0)}$, if and only if

$$\lim_{\nu\to\infty} (N^{-1}P)^{\nu} = 0. \tag{3.5}$$

This condition is satisfied provided some appropriate norm [3] of $(N^{-1}P)$, say the spectral norm, is <1.

This result is more frequently obtained by introducing the sequence of error vectors

$$E^{(\nu)} \equiv \Phi - \Phi^{(\nu)}, \tag{3.6}$$

which, by (3.1) and (3.3), must satisfy the homogeneous recursion

$$NE^{(\nu)} = PE^{(\nu-1)}, \qquad \nu \geq 1. \tag{3.7}$$

The eigenvalues, λ_k, of $N^{-1}P$ are the (pq) roots of the characteristic equation

$$| \lambda N - P | = 0. \tag{3.8}$$

If they are distinct* there then exists [4] a complete set of eigenvectors, e_k, satisfying

$$\lambda_k N e_k = P e_k, \tag{3.9}$$

which span the (pq)-dimensional vector space. Thus any initial error, $E^{(0)}$, has a unique expansion in these eigenvectors of the form

$$E^{(0)} = \sum_{k=1}^{pq} a_k e_k. \tag{3.10}$$

*It is sufficient here to assume that all the elementary divisors [4] of $(N^{-1}P)$ are simple.

By (3.7) and (3.9) the above yields, for the νth error vector,

$$E^{(\nu)} = \sum_{k=1}^{pq} \lambda_k^{\nu} a_k e_k . \tag{3.11}$$

In order that $\Phi^{(\nu)} \to \Phi$ it is necessary and sufficient that $E^{(\nu)} \to 0$. Thus by (3.11) and the completeness of the eigenvectors, convergence is equivalent, for arbitrary initial error*, to

$$\lambda_{\max} \equiv \max_k | \lambda_k | < 1. \tag{3.12}$$

Let us require that the most slowly decaying component in the νth error, (3.11), be reduced by at least 10^{-m}, where $m > 0$. Then we must have $\lambda_{\max}^{\nu} \leq 10^{-m}$ and the number of iterations required is bounded by

$$\nu \geq -m/\log \lambda_{\max} \equiv m/R. \tag{3.13}$$

This result is valid only when (3.12) is satisfied and then $R \equiv -(\log \lambda_{\max})^{-1}$ is called the rate of convergence. This quantity is useful in comparing different iterative methods as the number of iterations required for some specified convergence criterion varies as R^{-1}.

If the elementary divisors of $N^{-1}P$ are not simple, condition (3.12) still suffices for convergence but the bound (3.13) does not apply. To obtain such a bound we assume the elementary divisor of largest order corresponding to $\lambda_{\max} = | \lambda_1 |$ say, is of order $r + 1$. Then the expansion (3.10) must be replaced by [4]

$$E^{(\nu)} = \sum_{k=1}^{r+1} \left[\sum_{s=0}^{r+1-k} \binom{\nu}{s} \lambda_1^{\nu-s} a_{s+k} \right] e_k + \sum_{k=r+2}^{pq} \lambda_k^{\nu} a_k e_k \tag{3.14}$$

provided $\nu \geq (r + 1)$ (and assuming all other divisors to be simple). The largest decay factor is now given by, for $\nu \geq (r + 1)\lambda_{\max} + r$,

$$\lambda_{\max}^{\nu-r} \binom{\nu}{r};$$

and to reduce the error by at least 10^{-m} requires

$$\lambda_{\max}^{\nu-r} \binom{\nu}{r} \leq 10^{-m}.$$

An approximate bound on the number of iterations which may be obtained from this inequality, is

$$\nu \geq m'/R + r + (r/2R) \log [(m'/R + r)m'/R], \tag{3.15}$$

where $m' \equiv m - \log r!$. The number of iterations required for convergence is now not simply proportional to R^{-1} but this result is asymptotically (for $m \to \infty$) equivalent to (3.13). For practical computations the discrepancy may be significant.

A large rate of convergence should not be the only criterion in evaluating iterative methods. Rather a measure of the time required (by human and/or machine effort) to achieve the desired accuracy should be employed. This time is essentially proportional

*For a special initial error in which the amplitude of some particular component vanishes, say $a_1 = 0$, the corresponding eigenvalue, λ_1, is unrestricted. However, even if such initial distributions could be determined, computations with roundoff would undoubtedly introduce this component at an early stage of the iterations and it could become arbitrarily large if (3.12) is violated.

to the total number of arithmetic operations (property weighted). Thus if we let N_{op} be the operational count required for each iteration [i.e. to solve (3.3)], the total time is proportional to [assuming (3.13) to be valid],

$$T \equiv N_{op}/R. \tag{3.16}$$

The "best" of the single sweep methods is that for which T is minimized. Of course in more general procedures a similar criterion should be employed.

4. Special single sweep iterations and generalized extrapolation. With the above notions in mind we consider the special class of splittings

$$N(\gamma) \equiv \gamma_0 I - \gamma_1 L - \gamma_2 R - \gamma_3 B - \gamma_4 T, \qquad P(\gamma) \equiv N(\gamma) - M, \tag{4.0}$$

where the real numbers γ_i are restricted by the conditions that

$$\text{a)} \quad |\, N(\gamma) \,| \neq 0, \qquad \text{b)} \quad \gamma_1\gamma_2\gamma_3\gamma_4 = 0. \tag{4.1}$$

This defines a subset Γ of the five dimensional Euclidian space of all points γ. The iterations are defined by

$$N(\gamma)\Phi^{(\nu)} = P(\gamma)\Phi^{(\nu-1)} + S. \tag{4.2}$$

Thus any point $\gamma \in \Gamma$ determines an iterative scheme given by (4.0) and (4.2); we shall sometimes refer to this as the scheme γ. The class of schemes, Γ, is important since it includes many known and frequently used schemes while the data arrangement required for all of these schemes is well suited for automatic computing machines. Furthermore, the solution of the system (4.2) is obtained explicitly when $\gamma \in \Gamma$ in one sweep over the mesh by solving either two-term or three-term recursions. In particular if either $\gamma_0\gamma_1\gamma_2 \neq 0$ or $\gamma_0\gamma_3\gamma_4 \neq 0$ three-term recursions are introduced along horizontal or vertical mesh lines. The solution of these recursions may be reduced, by the well-known factorization of a Jacobi matrix, to the evaluation of two, two-term recursions along the appropriate lines. If two or more of the $\gamma_i \neq 0$ the method of sweeping the mesh to solve (4.2) in one sweep is partially determined (i.e. the sweep must start at a particular corner or with some line next to a boundary).

The operational counts, $N_{op}(\gamma)$, required to solve (4.2) for any of the schemes $\gamma \in \Gamma$ vary at most by a factor less than two. The minimum number of operations needed is four multiplications and five additions at each mesh point and the maximum required is seven multiplications and six additions (neglecting the operations done only once in factoring the matrix of Jacobi form). The special subclasses of Γ introduced in Sec. 6 require at most six multiplications and five additions at each point. Thus in the remainder of the paper we shall assume the operational counts to be almost equal and in seeking the "best" scheme we shall consider only the eigenvalues.

The eigenvalues of a scheme such as (4.0) to (4.2) are the roots λ of the characteristic equation

$$|\, \lambda N(\gamma) - P(\gamma) \,| = 0;$$

or explicitly, of the equation

$$\Delta \equiv |\, g_0 I - g_1 L - g_2 R - g_3 B - g_4 T \,| = 0, \tag{4.3}$$

where

$$g_i \equiv \gamma_i(\lambda - 1) + 1, \qquad 0 \leq i \leq 4.$$

Each root, $\lambda_k(\gamma)$, is thus a function of the scheme, γ, and a problem naturally suggested is to find a $\gamma^* \in \Gamma$ such that

$$\lambda_{\max}(\gamma^*) = \min_{\gamma \in \Gamma} (\max_k | \lambda_k(\gamma) |). \tag{4.4}$$

The solution of this problem would yield, essentially, the best iterative method of the class considered. The usual notion of extrapolation (or over-relaxation) of the scheme γ^* is meaningless, by virtue of (4.4), since no improvement on rate of convergence can be obtained. In fact the usual extrapolation techniques may be viewed as attempts to find the solution of (4.4) when γ is further restricted to lie on some particular curve in Γ. This is in fact shown to be the case in Sec. 6 and the curve, in the usual extrapolation procedures, is a straight line. The problem posed in (4.4) is thus a generalization of extrapolation in which the values of four extrapolation parameters are to be obtained [since, by (4.1b), Γ consists of four four-dimensional subspaces].

In order to obtain some idea of the possible behavior of $\lambda(\gamma)$ we consider schemes γ on the "diagonal" line

$$\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1/\alpha. \tag{4.5}$$

These points *do not* lie in Γ and, obviously, taking $\alpha = 1$ implies direct inversion of M. However, to gain insight, we use (4.5) in (4.3) and obtain the characteristic equation:

$$\left| \left[\frac{1}{\alpha} (\lambda - 1) + 1 \right] M \right| = 0.$$

Since, by assumption, $| M | \neq 0$, there is only one eigenvalue (with multiplicity pq) and it is

$$\lambda = 1 - \alpha.$$

Thus along the diagonal line (4.5) all schemes converge for $0 < \alpha < 2$ (i.e. $\gamma_i > 1/2$) and diverge otherwise (i.e. $\gamma_i < 1/2$). Only one iteration is required for $\alpha = 1$, as then $\lambda = 0$. It might be suspected that the best scheme in Γ is obtained by taking a point nearest this unit point [$\alpha = 1$ in (4.5)]. That this is not the case is a simple consequence of the results of Sec. 6. The eigenvalues of the above example become arbitrarily large in absolute value as $\alpha \to \pm\infty$ and the point γ approaches the origin; the roots approach unity as $\alpha \to \pm 0$ and the point γ recedes to infinity. It should also be observed that these eigenvalues are independent of the matrix M.

5. Some properties of the eigenvalues, $\lambda(\gamma)$. We present here some theorems which can be used to compare the eigenvalues of various schemes γ. All of these results are simple consequences of the

Fundamental theorem: Let L, R, B and T be arbitrary matrices of the form (2.6), (2.3). Then for any non-zero scalars x and y.

$$| M | \equiv | I - L - R - B - T | \equiv \left| I - xL - \frac{1}{x} R - yB - \frac{1}{y} T \right|. \tag{5.0}$$

Proof. Let the elements of M be $M_{r,s}$ where $1 \leq r, s \leq N = pq$. Then each term in the formal expansion of $| M |$ is given by [5]

$$\pm M_{1,\pi(1)} M_{2,\pi(2)} \cdots M_{r,\pi(r)} \cdots M_{N,\pi(N)}, \tag{5.1}$$

where π is one of the $N!$ permutations of the first N integers. Let each point (i, j) of

the original rectangular mesh (see Fig. 1) be identified with a unique integer $r \equiv (j-1)p + i$, and represent any permutation π by the N vectors from r to $\pi(r)$ on this mesh. By the definitions (2.6), (2.3) and (5.0), $M_{r,\pi(r)} \neq 0$ only if $\pi(r) = r$ or the point corresponding to $\pi(r)$ is an adjacent neighbor of the point r. Thus the only permutations which lead to non-vanishing terms (5.1) are those whose geometric representation is composed entirely of unit vectors in the $(\pm i)$- and $(\pm j)$-directions and null vectors. However, any permutation is a product of disjoint cycles [5] and the representation of a cycle is a closed path on the mesh. Thus for non-vanishing cycles there are the same number of unit vectors in the $(+i)$-direction as in the $(-i)$-direction and similarly for the $(\pm j)$-directions (see Fig. 1). Since $M_{r,\pi(r)}$ is an element of L if $\pi(r) = r - 1$, and an element of R if $\pi(r) = r + 1$, there are as many factors from L as from R in each non-vanishing cycle. A similar result is true of factors from B and T. Thus each non-vanishing term in the expansion of the right hand determinant in (5.0) is independent of x and y and the proof is complete.

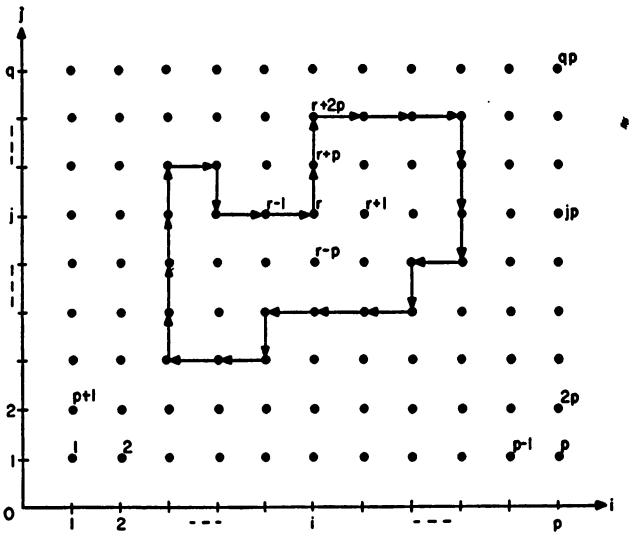

FIG. 1. Geometric representation of a non-vanishing cycle. In the permutation of, $n \to \pi(n)$, which this cycle is a factor the cycle is given by:

$$\pi(r) = r + p, \quad \pi(r + p) = r + 2p, \ldots\ldots, \quad \pi(r - 1) = r.$$

The geometric form of the above proof was suggested by D. Ludwig. Another proof has been given by B. Friedman [8]. However, the geometric proof clearly indicates that the same result can be proved for a matrix which represents the difference equations on any connected region bounded by coordinate segments. In higher dimensions the analogous theorem, obtained by adding two more matrices to M for each additional dimension, is easily proved. A special case of the above theorem, namely that obtained by setting $x = y$ in (5.0) has been proven by D. Young [1]. Many of the remaining results in this paper hold for any matrix M which can be written in the form (2.7) such that (5.0) is satisfied for arbitrary x and y.

The first, almost obvious, consequence of the fundamental theorem is contained in

Theorem I. Let $\lambda(\gamma) = \lambda(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be any eigenvalue of the scheme γ. Then for all γ:

$$\lambda(\gamma) = \lambda(\gamma_0, \gamma_2, \gamma_1, \gamma_3, \gamma_4) = \lambda(\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_3). \tag{5.2}$$

Proof. The eigenvalues $\lambda(\gamma)$ are the roots of the characteristic equation (4.3). However, by the fundamental theorem we have

$$\Delta = \left| g_0 I - x g_1 L - \frac{1}{x} g_2 R - y g_3 B - \frac{1}{y} g_4 T \right|,$$

and taking $x = (g_2/g_1)^{1/2}$, $y = (g_4/g_3)^{1/2}$ the above yields*

$$\Delta = | g_0 I - (g_1 g_2)^{1/2} (L + R) - (g_3 g_4)^{1/2} (B + T) |. \tag{5.3}$$

Thus the roots of $\Delta = 0$ are invariant under the interchanges $\gamma_1 \leftrightarrow \gamma_2$ and $\gamma_3 \leftrightarrow \gamma_4$.

In terms of iterative procedures of the class (4.2), Theorem I indicates that various ways of sweeping the mesh yield identical rates of convergence. In terms of the subspace Γ the theorem states that there are certain two-dimensional planes with respect to which the eigenvalues are symmetric. Thus the volume of Γ which must be searched for a best scheme, γ^*, is reduced by a factor of 1/4. Furthermore it is of interest to note that along the two-dimensional planes $\gamma_1 = \gamma_2$ and $\gamma_3 = \gamma_4$ the eigenvalues $\lambda(\gamma)$ must have relative maxima or minima with respect to the appropriate pair of coordinates.

A somewhat more general result which contains Theorem I as a special case is contained in

Theorem II. Let $\lambda'(\gamma')$ be some particular eigenvalue of the scheme γ'. For any scheme γ let $\lambda(\gamma)$ be a function such that

$$\left(\frac{g_0}{g_0'}\right)^2 = \frac{g_1 g_2}{g_1' g_2'} = \frac{g_3 g_4}{g_3' g_4'} \equiv \xi^2 \neq 0, \tag{5.4}$$

where g_i' and g_i are respectively the functions of (λ', γ') and (λ, γ) defined in (4.3). Then $\lambda(\gamma)$ is an eigenvalue of the scheme γ.

Proof. Form the determinant Δ of (4.3). Then as in the proof of Theorem I we obtain (5.3). Using (5.4) this becomes

$$\Delta = \xi^{pq} | g_0' I - (g_1' g_2')^{1/2}(L + R) - (g_3' g_4')^{1/2}(B + T) | = \xi^{pq} \Delta',$$

by another application of the fundamental theorem to the scheme γ'. However, since $\lambda'(\gamma')$ is an eigenvalue of γ' we have $\Delta' = 0$. Thus $\Delta = 0$ and $\lambda(\gamma)$ must be an eigenvalue of γ.

The schemes γ and γ', and the eigenvalues $\lambda(\gamma)$ and $\lambda'(\gamma')$ of the theorem will be called images of each other. We sometimes refer to γ' or $\lambda'(\gamma')$ as the reference scheme or reference eigenvalue respectively. It is clear from (5.4) that the relationship of being images is a transitive one. Some consequences of Theorem II are examined in the remaining sections.

Another theorem which is quite useful for some special difference equations is

Theorem III. Let the matrices (2.6) be such that $(L + R)(B + T) = (B + T)(L + R)$. Then as is well known these matrices have common eigenvectors; let ρ_k and μ_k be the eigenvalues of $(L + R)$ and $(B + T)$ respectively, corresponding to the common eigenvector e_k. Then each eigenvalue of any scheme γ is a root, for some k, of the (at most 4th degree) equation**

$$g_0 - (g_1 g_2)^{1/2} \rho_k - (g_3 g_4)^{1/2} \mu_k = 0. \tag{5.5}$$

*The chosen branches of the square roots are arbitrary and hence all future applications of (5.3) could be written in any of four ways.

**As pointed out in the proof of Theorem I there are four such equations which could be used. This point is clarified in Sec. 7 where more of the consequences of the hypothesis are examined.

Proof. Let $\lambda(\gamma)$ be some fixed but as yet unspecified eigenvalue of the scheme γ. Then from (4.3), $\Delta = 0$, and as in the proof of Theorem I, we have from (5.3)

$$| g_0 I - (g_1 g_2)^{1/2}(L + R) - (g_3 g_4)^{1/2}(B + T) | = 0. \tag{5.6}$$

Thus the matrix in (5.6) is singular and has a zero eigenvalue. Applying this matrix to e_k we see that

$$\xi_k \equiv g_0 - (g_1 g_2)^{1/2}\rho_k - (g_3 g_4)^{1/2}\mu_k$$

is an eigenvalue belonging to the eigenvector e_k . Using all the e_k we obtain all of the eigenvalues of the matrix in (5.6). However at least one $\xi_k = 0$. Since $\lambda(\gamma)$ was any eigenvalue of the scheme γ the theorem follows.

This theorem is applied generally in Sec. 7 and to the usual Laplace difference equations in Sec. 8.

6. The complete image classes. The pairs of image schemes defined by Theorem II are such that only one eigenvalue of any γ need be the image of one eigenvalue of the reference scheme, γ'. However, of special interest are those schemes each of whose eigenvalues is the image of a corresponding eigenvalue of some particular reference scheme. Such classes of schemes are called complete image classes and we proceed to obtain three of them. The intersections of these classes with the subspace Γ is grouped into five subspaces: Γ_A , Γ_B , $\Gamma_{B'}$, Γ_C and $\Gamma_{C'}$, of which $\Gamma_{B'}$ and $\Gamma_{C'}$ are considered uninteresting.

The class Γ_A . In order that any two schemes γ and γ' be images, (5.4) must be satisfied by the corresponding pair of image eigenvalues. However, let us seek first schemes such that

$$\frac{g_1 g_2}{g_1' g_2'} \equiv \frac{g_3 g_4}{g_3' g_4'} \tag{6.0}$$

is an identity in the indeterminates λ and λ'. This requires

$$\text{a)} \begin{cases} \gamma_1\gamma_2 = \gamma_3\gamma_4 \\ \gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 , \end{cases} \qquad \text{b)} \begin{cases} \gamma_1'\gamma_2' = \gamma_3'\gamma_4' \\ \gamma_1' + \gamma_2' = \gamma_3' + \gamma_4' . \end{cases} \tag{6.1}$$

Now let λ' be any eigenvalue of a scheme γ' which satisfies (6.1b). Then if γ is any scheme satisfying (6.1a) every root λ of

$$\left(\frac{g_0}{g_0'}\right)^2 = \frac{g_1 g_2}{g_1' g_2'} \tag{6.2}$$

is an eigenvalue of γ by Theorem II. Since (6.0) is satisfied for any λ and λ', each equation (6.2) obtained for a different eigenvalue λ' of γ' determines at least one eigenvalue λ of the scheme γ. Thus all schemes γ satisfying (6.1a) belong to the same complete image class, say class A. Since the conditions (6.1a) and (6.1b) are identical any $\gamma \in A$ may be used as the particular reference scheme. We call the chosen reference scheme γ_A and take it to be

$$\gamma_A : \gamma_0 = 1, \qquad \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0.$$

This is a simultaneous displacement method commonly known as Richardson's method. Thus each of the eigenvalues of any class A scheme is the image of one of the Richardson eigenvalues.

The class A schemes of interest are those contained in Γ. From (4.1) and (6.1) we obtain the set of all such schemes, Γ_A, and they are listed in Table A. The real parameters α and β are to be restricted such that (4.1a) is satisfied. The set Γ_A lies on four two-dimensional planes in Γ, one of which is, say: $\gamma_2 = \gamma_4 = 0, \gamma_1 = \gamma_3$. By the symmetry

	γ_0	γ_1	γ_2	γ_3	γ_4
Γ_A	$1/\alpha$	0	$1/\beta$	$1/\beta$	0
	$1/\alpha$	$1/\beta$	0	$1/\beta$	0
	$1/\alpha$	0	$1/\beta$	0	$1/\beta$
	$1/\alpha$	$1/\beta$	0	0	$1/\beta$
γ_A	1	0	0	0	0

TABLE A

property of the eigenvalues, expressed in Theorem I, we need examine only one of these four planes. However using any $\gamma \in \Gamma_A$ and γ_A in (6.2) yields

$$[\lambda - (1 - \alpha)]^2 = \frac{\alpha^2}{\beta} \lambda_A^2 [\lambda - (1 - \beta)]. \tag{6.3}$$

This equation furnishes the mapping of the eigenvalues λ_A of γ_A (Richardson) onto the eigenvalues λ of any of the schemes in Γ_A.

If we choose $\alpha = \beta = 1$ in any $\gamma \in \Gamma_A$ the resulting scheme is a successive displacement method commonly known as a Liebmann scheme, in the present connection, or more generally as the Gauss-Seidel method (see Table I). The mapping (6.3) yields

$$\lambda = \lambda_A^2, \quad \text{or} \quad \lambda = 0. \tag{6.4}$$

Thus some λ_A go into zero and others into their squares. These schemes converge, as is well known [7], if the Richardson scheme converges, and the rate of convergence is twice as large (see Sec. 3).

If we set $\beta = 1$ in any $\gamma \in \Gamma_A$ the resulting scheme is a successive overrelaxation [1] method where α is the overrelaxation parameter (see Table I); this method is frequently called the extrapolated Liebmann scheme [6]. The schemes γ, in this case, lie along four lines in Γ_A, for example: $\gamma_2 = \gamma_4 = 0, \gamma_1 = \gamma_3 = 1$. The mapping (6.3) becomes

$$[\lambda - 1 + \alpha]^2 = \alpha^2 \lambda_A^2 \lambda, \tag{6.5}$$

which has been studied thoroughly [1].

Taking $\alpha = \beta$ in any $\gamma \in \Gamma_A$ yields a different successive overrelaxation method; this corresponds to using successive displacements (Liebmann) over the entire mesh and then extrapolating (or interpolating) the provisional new iterate with the old iterate. The mapping (6.3) becomes, in this case,

$$\lambda = 1 - \alpha(1 - \lambda_A^2), \tag{6.6}$$

TABLE I

Some standard methods as complete images.

Name, Type and Class	γ_0	γ_1	γ_2	γ_3	γ_4	g_0	g_1	g_2	g_3	g_4	$\left(\frac{g_0}{g_0'}\right)^2$	$\frac{g_1 g_2}{g_1' g_2'}$	$\frac{g_3 g_4}{g_3' g_4'}$
Richardson *A* (Simultaneous displacement)	1	0	0	0	0	ρ	1	1	1	1	...	...	...
Liebmann (*G-S*) *A* (Successive displacement)	1	1	0	1	0	λ	λ	1	λ	1	$(\lambda/\rho)^2$	λ	λ
Extrapolated Liebmann *A* (Successive overrelaxation)	$\frac{1}{\alpha}$	1	0	1	0	$\frac{\xi - 1 + \alpha}{\alpha}$	ξ	1	ξ	1	$\left(\frac{\xi - 1 + \alpha}{\alpha\rho}\right)^2$ $\left(\frac{\xi - 1 + \alpha}{\alpha\lambda}\right)^2$	ξ ξ/λ	ξ ξ/λ
Line-Richardson *B* (Group simultaneous displacement)	1	1	1	0	0	ν	ν	ν	1	1	...	...	...
Line-Liebmann *B* (Group successive displacement)	1	1	1	1	0	ρ	ρ	ρ	ρ	1	$(\rho/\nu)^2$	$(\rho/\nu)^2$	ρ
Extrapolated Line-Liebmann (Group successive overrelaxation) *B*	$\frac{1}{\alpha}$	$\frac{1}{\alpha}$	$\frac{1}{\alpha}$	1	0	$\frac{\mu - 1 + \alpha}{\alpha}$	$\frac{\mu - 1 + \alpha}{\alpha}$	$\frac{\mu - 1 + \alpha}{\alpha}$	μ	1	$\left(\frac{\mu - 1 + \alpha}{\alpha\nu}\right)^2$ $\left(\frac{\mu - 1 + \alpha}{\alpha\rho}\right)^2$	$\left(\frac{\mu - 1 + \alpha}{\alpha\nu}\right)^2$ $\left(\frac{\mu - 1 + \alpha}{\alpha\mu}\right)^2$	μ μ/ρ

which is easily analyzed. If the λ_A are all real and $\lambda_{\max,A} < 1$, then $\lambda_{\max}$ is minimized by taking

$$\alpha = [1 - \tfrac{1}{2}(\lambda^2_{\max,A} + \lambda^2_{\min,A})]^{-1},$$

where $\lambda_{\min,A} \equiv \min_k | \lambda_k(\gamma_A) |$. This method is the analogue of the case treated at the end of Sec. 4; it is called a full-mesh extrapolation.

If all of the eigenvalues λ_A are real and $\lambda_{\max,A} < 1$ the general mapping (6.3) can be analyzed. It is found, in this case, that the best of the class A schemes is extrapolated Liebmann. However, in other cases, it seems possible to improve upon the Liebmann extrapolations. The analysis of this mapping has been done by K. Gordis and will be reported in a future paper.

The class Γ_B . Proceeding as in the previous case we now require

$$\left(\frac{g_0}{g_0'}\right)^2 \equiv \frac{g_1 g_2}{g_1' g_2'}$$

to be an identity in λ and λ'. This is equivalent to

$$\text{a)}\quad \gamma_0 = \gamma_1 = \gamma_2\,, \qquad \text{b)}\quad \gamma_0' = \gamma_1' = \gamma_2'\,. \tag{6.7}$$

Using (4.1b) these relations yield the indicated classes Γ_B and $\Gamma_{B'}$ of Table B.

	γ_0	γ_1	γ_2	γ_3	γ_4
Γ_B	$1/\alpha$	$1/\alpha$	$1/\alpha$	$1/\beta$	0
	$1/\alpha$	$1/\alpha$	$1/\alpha$	0	$1/\beta$
Γ_B'	0	0	0	$1/\alpha$	$1/\beta$
γ_B	1	1	1	0	0

TABLE B

The canonical reference scheme, γ_B , included in the table is a simultaneous line-displacement method; by its analogy with γ_A we shall call it a line-Richardson method. The classes of schemes Γ_B and $\Gamma_{B'}$ lie on three two-dimensional planes in Γ. As before if we take λ_B to be any eigenvalue of γ_B then by Theorem II any root λ of

$$\frac{g_3 g_4}{g_3' g_4'} = \left(\frac{g_0}{g_0'}\right)^2$$

is an eigenvalue of $\gamma \in \Gamma_B$ (or $\Gamma_{B'}$) of the table. Of course the appropriate $\gamma \in \Gamma_B$ (or $\Gamma_{B'}$) is to be used in the g_i . Using $\gamma \in \Gamma_B$ and γ_B of the table this yields

$$[\lambda - (1 - \alpha)]^2 = \frac{\alpha^2}{\beta}\lambda_B^2[\lambda - (1 - \beta)]; \tag{6.8}$$

the same mapping (6.3) as in the Γ_A schemes. However, the reference schemes γ_A and γ_B are not images of each other and thus we cannot, in general compare their eigenvalues. In Secs. 7 and 8 special cases are considered in which λ_A and λ_B may be compared; it is then clear that some schemes in Γ_B are better than the best scheme in Γ_A .

All of the simplifications (6.4–.6) are shown to apply for $\gamma \in \Gamma_B$ by choosing the

same special α and β as were chosen for $\gamma \in \Gamma_A$ (see Table I). Thus we may call $\gamma = (1, 1, 1, 1, 0)$ "line-Liebmann", and $\gamma = (1/\alpha, 1/\alpha, 1/\alpha, 1, 0)$ "extrapolated line-Liebmann" and their eigenvalues have exactly the same relationships to the line-Richardson eigenvalues as the eigenvalues of ordinary Liebmann and extrapolated Liebmann have to those of ordinary Richardson.

The remaining class of schemes, $\Gamma_{B'}$, yield the mapping

$$[\lambda - (1 - \alpha)][\lambda - (1 - \beta)] = \alpha\beta\lambda_B^2 . \tag{6.9}$$

This result has been examined and is in general found to be inferior to that of (6.8).

The class Γ_C. This class is determined exactly as the previous one by interchanging (γ_1, γ_2) with (γ_3, γ_4). The mappings (6.8) and (6.9) are obtained for the corresponding classes Γ_C and $\Gamma_{C'}$. The canonical reference scheme is $\gamma_C = (1, 0, 0, 1, 1)$, a line-Richardson method, which is not an image of γ_A or γ_B. In the special cases of the next two sections it is possible to compare λ_B and λ_C and thus to determine which of the classes, Γ_B or Γ_C contains the best scheme.

The essential difference between Γ_B and Γ_C schemes is the direction of the line along which three-term recursions must be solved. In all Γ_B schemes they are parallel to the direction of increasing i and in Γ_C parallel to the direction in which j increases.

7. Reference eigenvalues for a special class of equations. We consider here those systems of the form (2.2) to (2.7) for which $(L + R)$ and $(B + T)$ commute. Then the fundamental theorem and Theorem III are valid. However, before applying these results we shall examine some other very important consequences of commutativity for matrices of the indicated forms.

By using the forms (2.6) in

$$(L + R)(B + T) = (B + T)(L + R)$$

we obtain the conditions

$$\begin{aligned} (L_j + R_j)B_j &= B_j(L_{j-1} + R_{j-1}), \qquad 2 \le j \le q, \\ (L_j + R_j)T_j &= T_j(L_{j+1} + R_{j+1}), \qquad 1 \le j \le q - 1. \end{aligned}$$

These conditions are necessary and sufficient for commutativity. With no loss in generality* we may assume $|B_j| \neq 0$ and $|T_j| \neq 0$ in which case the above conditions become

$$(L_j + R_j) = \begin{cases} B_j(L_{j-1} + R_{j-1})B_j^{-1}, & 2 \le j \le q, \\ T_j(L_{j+1} + R_{j+1})T_j^{-1}, & 1 \le j \le q - 1. \end{cases} \tag{7.0}$$

Since the $(L_j + R_j)$ are all similar they have the same eigenvalues and elementary divisors; and as the order of these matrices is $p \times p$ there are at most p distinct eigenvalues ρ_i, $1 \le i \le p$. From the form (2.6) of $(L + R)$ we see that its eigenvalues are just the ρ_i, each of whose multiplicity has a factor q [i.e. if ρ_i has multiplicity r_i in $(L_j + R_j)$ then it has multiplicity $r_i q$ in $(L + R)$]. If the ρ_i are all distinct (or belong to simple elementary divisors) it can be shown from (7.0) that for some scalars a_j,

$$B_{j+1} = a_j T_j^{-1}, \qquad 1 \le j \le q - 1. \tag{7.1}$$

*Only the boundary conditions may introduce singular B_j and T_j for elliptic difference equations. However, these cases may be eliminated, depending on the boundary conditions, by either not including the boundary values as unknowns (as in Sec. 8) or by requiring the difference analogue of the equations to hold at boundary points.

By changing the ordering (2.2), (2.5) from the original row-ordering to a column-ordering (which is equivalent to a similarity transformation of L, R, B and T) we find, as above, that the eigenvalues μ_i of $(B + T)$ are the eigenvalues of p similar $q \times q$ matrices. As eigenvalues of $(B + T)$ each μ_i has a multiplicity which is a multiple of p and there are at most q such eigenvalues. It is well known that commuting matrices have common eigenvectors. Thus in the above case there exist common eigenvectors e_{ij} such that

$$(L + R)e_{ij} = \rho_i e_{ij} , \qquad (B + T)e_{ij} = \mu_j e_{ij} , \tag{7.2}$$

for each ρ_i and μ_j belonging to different elementary divisors of $(L + R)$ and $(B + T)$ respectively.

To determine some properties of the ρ_i and μ_j we consider the characteristic equations of which they are the roots. Exactly as in the proof of the fundamental theorem we see that

$$| \rho I - (L_i + R_i) | = \left| \rho I - xL_i - \frac{1}{x} R_i \right|$$

for arbitrary $x \neq 0$. Thus taking $x = -1$ we find that if ρ is an eigenvalue of $(L_i + R_i)$ then so is $-\rho$. Furthermore if p is odd there is at least one zero eigenvalue and additional zero eigenvalues must occur in pairs. Analogous results* are true of the eigenvalues μ of $(B + T)$.

We are now in a position to apply Theorem III. We select first $\gamma_A \equiv (1, 0, 0, 0, 0)$ and using (7.2) we may replace the pair (ρ_k , μ_k) of the theorem by the pair (ρ_i , μ_j) to obtain from (5.5)

$$\lambda_{ij,A} = \rho_i + \mu_j . \tag{7.3A}$$

Similarly using $\gamma_B \equiv (1, 1, 1, 0, 0)$ we get

$$\lambda_{ij,B} = \frac{\mu_j}{1 - \rho_i} , \tag{7.3B}$$

and, finally, with $\gamma_C = (1, 0, 0, 1, 1)$ Theorem III yields

$$\lambda_{ij,C} = \frac{\rho_i}{1 - \mu_j} . \tag{7.3C}$$

We have introduced an obvious double subscript notation for the λ. These are all of the roots of the corresponding schemes since (7.3) holds for all ρ_i and μ_j . Using other schemes γ in (5.5) would yield many other eigenvalues explicitly. However, any of the class Γ_A , Γ_B or Γ_C scheme eigenvalues are obtained by using (7.3) in (6.2) etc.

As a further condition let us assume that the ρ_i and μ_j are all real. (A sufficient condition for this would be that $(L + R)$ and $(B + T)$ are symmetric. Then $B_{i+1} = T_i$ and by (7.1) the T_i and B_i must be constants times the identity. Similar results would hold for the transformed $(L + R)$ block matrices.) Then by the parity of the eigenvalues ρ_i and μ_j (i.e. since $\max_i | \rho_i | = \max_i \rho_i$) we obtain from (7.3A)

$$\lambda_{\max,A} = \rho_{\max} + \mu_{\max} . \tag{7.4A}$$

*The properties of the eigenvectors (7.2) and the sign parity of the eigenvalues explains the apparent ambiguity pointed out in Theorem III, in which any of four seemingly different equations could have been used. They are not different but correspond to different labeling of the eigenvalues.

Furthermore, if $\rho_{\max} < 1$ and $\mu_{\max} < 1$ we get from (7.3B, C):

$$\lambda_{\max,B} = \frac{\mu_{\max}}{1 - \rho_{\max}}; \tag{7.4B}$$

$$\lambda_{\max,C} = \frac{\rho_{\max}}{1 - \mu_{\max}}. \tag{7.4C}$$

Thus we may easily compare these principal eigenvalues to obtain

$$\begin{aligned} &\text{a)} \quad \rho_{\max} + \mu_{\max} \lesseqgtr 1 \Rightarrow \lambda_{\max,A} \gtreqless (\lambda_{\max,B}, \lambda_{\max,C}), \\ &\text{b)} \quad |\rho_{\max} - 1/2| \lesseqgtr |\mu_{\max} - 1/2| \Rightarrow \lambda_{\max,B} \lesseqgtr \lambda_{\max,C}. \end{aligned} \tag{7.5}$$

Since the mappings of the eigenvalues of the schemes in Γ_A, Γ_B and Γ_C from the corresponding image scheme eigenvalues are the same, the best scheme will be found in that class for which $\lambda_{\max}$ is smallest. The relations (7.5) may thus be used to determine the best class. It should be noted, from (7.5a), that in the present case if the Richardson method converges ($\lambda_{\max,A} < 1$) then the line-Richardson methods converge faster, and if Richardson does not converge neither do the line-Richardson methods.

8. Laplace's equation. As an example we consider $\Delta^2\phi = 0$ in the rectangle $0 \le x \le 1, 0 \le y \le 1$; with $\phi =$ (given function) on the boundary. Using the mesh

$$\begin{aligned} x_i &= i\Delta x, \qquad \Delta x \equiv \frac{1}{p}, \qquad 0 \le i \le p; \\ y_j &= j\Delta y, \qquad \Delta y \equiv \frac{1}{q}, \qquad 0 \le j \le q; \end{aligned} \tag{8.0}$$

and the difference approximations $\partial^2\phi/\partial x^2 \doteq 1/\Delta x^2\,(\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j})$, etc., at (x_i, y_j), we obtain the difference equations

$$\phi_{ij} = \theta_x[\phi_{i-1,j} + \phi_{i+1,j}] + \theta_y[\phi_{i,j-1} + \phi_{i,j+1}], \qquad \begin{cases} 1 \le i \le p-1 \\ 1 \le j \le q-1 \end{cases}. \tag{8.1}$$

Here $\phi_{0,j}$, $\phi_{p,j}$, $\phi_{i,0}$ and $\phi_{i,q}$ are the given boundary values and

$$\theta_x \equiv \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}, \qquad \theta_y \equiv \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}, \qquad (\theta_x + \theta_y = \tfrac{1}{2}). \tag{8.2}$$

Equations (8.1) may be written in the notation of Sec. 2 where $l_{ij} = r_{ij} = \theta_x$, $b_{ij} = t_{ij} = \theta_y$ and the inhomogeneous terms come from the boundary values; the system is of order $[(p-1)(q-1)]^2$. The matrices $(L + R)$ and $(B + T)$ then commute and are symmetric, and as proved in Sec. 7, their eigenvalues are the same as those of respectively,

$$\theta_x \begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & 1 \\ 0 & & & & 1 & 0 \end{bmatrix} \quad \text{and} \quad \theta_y \begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & 1 \\ 0 & & & & 1 & 0 \end{bmatrix}.$$

$(p-1) \times (p-1)$ $\qquad$ $(q-1) \times (q-1)$

The eigenvalues of these matrices are easily found to be

$$\begin{aligned} \rho_i &= 2\theta_x \cos(i\pi/p), \qquad 1 \le i \le p-1; \\ \mu_j &= 2\theta_y \cos(j\pi/q), \qquad 1 \le j \le q-1. \end{aligned} \tag{8.3}$$

The parity properties deduced in Sec. 7 are seen to be satisfied as $\rho_i = -\rho_{p-i}$ and $\mu_j = -\mu_{q-j}$. Thus the principal eigenvalues are obtained for $i = j = 1$, and we have

$$\begin{aligned} \rho_{\max} &= 2\theta_x \cos \pi/p \approx 2\theta_x - \theta_x(\pi/p)^2, \\ \mu_{\max} &= 2\theta_y \cos \pi/q \approx 2\theta_y - \theta_y(\pi/q)^2, \end{aligned} \tag{8.4}$$

where the approximate values hold for $p \gg \pi$ and $q \gg \pi$.

Let us consider first iterative solutions of (8.1) by means of the canonical reference schemes of Sec. 6 (all of which are simultaneous displacement methods). The principal eigenvalues of these schemes are obtained, since Theorem III applies, by using (8.4) in (7.3); then, as above, retaining only terms up to second order in $1/p$ and $1/q$ we have

$$\begin{aligned} \lambda_{\max,A} &\approx 1 - \pi^2\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Richardson]} \\ \lambda_{\max,B} &\approx 1 - \frac{\pi^2}{2\theta_x}\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Horizontal line-Richardson]} \\ \lambda_{\max,C} &\approx 1 - \frac{\pi^2}{2\theta_y}\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Vertical line-Richardson].} \end{aligned} \tag{8.5}$$

In this approximation we see that the direction of sweep, for minimum $\lambda_{\max}$ of the above line-methods, is determined only by the mesh ratio and does not depend upon the number of mesh points in the x or y-directions (since both p and q were assumed large). If the number of points is "small" the analogous criterion is obtained by using the exact eigenvalues (8.4) in (7.5). The rates of convergence (Sec. 3) of the above schemes are easily compared by recalling that $-\log(1-\epsilon) \approx \epsilon$ for small ϵ. If the mesh is square $\theta_x = \theta_y = 1/4$ and the line methods converge twice as fast as the ordinary Richardson (to second order in $1/p$). If the mesh is rectangular then $\theta_x < 1/4$ or $\theta_y < 1/4$ and by sweeping in the proper direction further improvement is obtained (the implicit equations should come from lines parallel to the direction of largest mesh spacing).

The successive displacement methods corresponding to the above reference schemes may be taken as (see Table I)

$$\gamma_a = (1, 1, 0, 1, 0), \qquad \gamma_b = (1, 1, 1, 1, 0), \qquad \gamma_c = (1, 1, 0, 1, 1). \tag{8.6}$$

The eigenvalues of these schemes are related to the corresponding reference eigenvalues (since they are complete image schemes) by (6.3) with $\alpha = \beta = 1$. We note as in Sec. 6 that $\lambda = 0$ may become an eigenvalue of the schemes (8.6) independently of the values of λ'. The remaining roots become $\lambda = (\lambda')^2$, and corresponding to (8.5) we get, up to second order in $1/p$ and $1/q$,

$$\begin{aligned} \lambda_{\max,a} &\approx 1 - 2\pi^2\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Liebmann]} \\ \lambda_{\max,b} &\approx 1 - \frac{\pi^2}{\theta_x}\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Horizontal line-Liebmann]} \\ \lambda_{\max,c} &\approx 1 - \frac{\pi^2}{\theta_y}\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Vertical line-Liebmann].} \end{aligned} \tag{8.7}$$

Of course the rates of convergence are now rigorously twice those of the reference scheme rates. The best sweep direction is determined as in the previous case, and again the best line method converges at least twice as fast as the ordinary Liebmann (to second order in $1/p$ and $1/q$).

The successive overrelaxation schemes which are usually applied to the above are

$$\gamma_a = (1/\alpha, 1, 0, 1, 0), \quad \gamma_b = (1/\alpha, 1/\alpha, 1/\alpha, 1, 0), \quad \gamma_c = (1/\alpha, 1, 0, 1/\alpha, 1/\alpha), \tag{8.8}$$

and now (6.3) holds in each case with $\beta = 1$. As is well known [1] for γ_a above (which is extrapolated Liebmann) and hence for all these cases, $\lambda_{\max}$ will be a minimum when α is chosen as the smaller root of

$$\lambda_{\max,X}^2 \alpha^2 - 4\alpha + 4 = 0, \tag{8.9A}$$

where $X = A$, B or C.
[That is, solve (6.3) for λ and set the discriminant equal to zero when $\lambda_A = \lambda_{\max,A}$.]
Then we have all $|\lambda| = \lambda_{\max}$ where

$$\lambda_{\max} = \alpha - 1. \tag{8.9B}$$

Using the reference eigenvalues (8.5) in the above we obtain for the schemes (8.8), up to second order in $1/p$ and $1/q$,

$$\begin{aligned}
\lambda_{\max,a} &= 1 - 2\pi\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right)^{1/2} + 2\pi^2\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Extrapolated Liebmann]} \\
\lambda_{\max,b} &= 1 - (2/\theta_x)^{1/2}\pi\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right)^{1/2} + \frac{\pi^2}{\theta_x}\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right), && \text{[Extrapolated horizontal line-Liebmann]} \\
\lambda_{\max,c} &= 1 - (2/\theta_y)^{1/2}\pi\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right)^{1/2} + \frac{\pi^2}{\theta_y}\left(\frac{\theta_x}{p^2} + \frac{\theta_y}{q^2}\right). && \text{[Extrapolated vertical line-Liebmann].}
\end{aligned} \tag{8.10}$$

There is an order of magnitude improvement in the rates of convergence of the extrapolated schemes over the previous schemes. That is, since $(\theta_x/p^2 + \theta_y/q^2) \equiv \Delta x^2 \Delta y^2 (\Delta x^2 + \Delta y^2)^{-1}$, the present rates are $O(\Delta x)$ or $O(\Delta y)$ while from (8.5) and (8.7) the rates are $\theta(\Delta x^2)$ or $\theta(\Delta y^2)$. The line schemes now improve the convergence rate by a factor $2^{1/2}$ for a square mesh or larger for rectangular meshes swept in the proper direction.

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