We next consider evaluation of the following infinite series (see[4]):

$$S = \sum_{s=1}^{\infty} k_s x_s ,$$

where

$$k_1^{-1} = 2.33, \qquad k_2^{-1} = 4.22, \qquad k_3^{-1} = 5.51, \qquad k_4^{-1} = 6.55,$$

and $k_s^{-1} \sim (2\pi)^{1/2}(2s - 1)^{1/2}$ for large s. If we evaluate

$$S_n = \sum_{s=1}^{n} k_s x_s^{(n)}$$

we obtain the following results:

$$S_1 = 0.155_5 : \qquad S_2 = 0.166_1 : \qquad S_3 = 0.169_1 : \qquad S_4 = 0.170_5 .$$

These underestimate S. In order to obtain overestimates of S we evaluate, using the previous approximation (5),

$$S_n' = \sum_{s=1}^{n} k_s x_s^{(n)} + k_n x_n^{(n)}(2n - 1)^2 \sum_{s=n+1}^{\infty} \frac{1}{(2s - 1)^2}$$
$$\approx \sum_{s=1}^{n} k_s x_s^{(n)} + k_n x_n^{(n)}(2n - 1)^2(4n)^{-1},$$

where we have replaced summation by integration as before. This gives

$$S_1' = 0.194_4 : \qquad S_2' = 0.181_5 : \qquad S_3' = 0.178_5 : \qquad S_4' = 0.176_9 .$$

The mean values $\frac{1}{2}(S_n + S_n')$ are remarkably constant. We can say from these results that $0.171 < S < 0.177$ with a probable value of $S = 0.174$.

ON SIMPLE SUBHARMONICS*

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Introduction. In a recent paper [1],† Rosenberg clarified considerably the curious subharmonic phenomena of non-linear oscillations by introducing the concepts of strong subharmonic solutions and of the simple subharmonics. He considered a system with a single degree of freedom, whose mechanical model might be a mass under the action of an elastic force, linear or non-linear, and of a simple harmonic forcing function of frequencey ω. The equation of motion of the system can be put in the form

$$\frac{d^2x}{dt^2} + f(x) = P_0 \cos \omega t, \tag{1}$$

where P_0 is proportional to the amplitude of the external forcing function.

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†Numbers in square brackets refer to the bibliography at the end of the paper.

If we are concerned only with the steady periodic solutions of (1), we can express them in terms of Fourier expansions. If such an expansion contains a term $A_r \cos(\omega_r t + \varphi_r)$, where $\omega_r = \omega/r$ and r is an integer, and if $A_r \neq 0$, the solution is said to be subharmonic of order $1/r$, and we have

$$x = A_r \cos\left(\frac{\omega}{r}\, t + \varphi_r\right) + \sum_{i \neq r} A_i \cos(\omega_i t + \varphi_i). \tag{2}$$

Rosenberg called the solution a *strong* subharmonic of order $1/r$, if $| A_r | \gg | A_i |$ for all i. He called the solution a *pure* subharmonic if $A_r \neq 0$ and $A_i = 0$ for all i. A solution is said to be a *simple* subharmonic if it is a pure subharmonic with zero phase angle φ_r, i.e. $x = x_0 \cos(\omega t/r)$.

Rosenberg showed that, for a class of differential equations of the following type

$$\frac{d^2\xi}{d\tau^2} + r^{-2}\xi + k \sum_{n}^{r} \alpha_r^{(n)}\xi^n = k \cos \tau \qquad (r = 1, 2, 3, \cdots), \tag{3}$$

periodic solutions are of the simple subharmonic type

$$\xi = \cos \tau/r. \tag{4}$$

Here a set of new variables has been introduced. In terms of the original variables they are: $\tau = \omega t$, $\xi = x/x_0$ and $k = P_0/(x_0\omega^2)$. A table was included in [1] giving the values of $\alpha_r^{(n)}$ in (3). The linear problem is a special case of (3) and (4). Then Eqs. (3) and (4) respectively become

$$\frac{d^2\xi}{d\tau^2} + (1 + k)\xi = k \cos \tau, \tag{5}$$

and

$$\xi = \cos \tau. \tag{6}$$

This is the usual linear response solution.

In the present note the writer will try to amplify and discuss some aspects of this interesting subharmonic problem. It will be shown that by a single transformation all of the equations (3) and their subharmonic solutions (4) can be derived from the linear case (5) and (6).

Tchebycheff polynomials. A careful study of Table 1 of [1] which gave the values of $\alpha_r^{(n)}$ indicates that $\alpha_r^{(n)}$ are simply the coefficients of various Tchebycheff polynomials of the first kind. This was also mentioned in another paper [2] by Rosenberg. Thus Eq. (3) can be written as

$$\frac{d^2\xi}{d\tau^2} + r^{-2}\xi + kT_r(\xi) = k \cos \tau \tag{7}$$

and its periodic solution given by the simple subharmonic (4). For ease of reference the Tchebycheff polynomials of the first kind will be briefly described here. The simplest definition is as follows. If we put $y = \cos \theta$, the Tchebycheff polynomial of the first kind of order r, denoted by $T_r(y)$, is defined as

$$T_r(y) = \cos r\theta. \tag{8}$$

The first few of these are:

$$\begin{aligned} T_0(y) &= 1, & T_3(y) &= 4y^3 - 3y, \\ T_1(y) &= y, & T_4(y) &= 8y^4 - 8y + 1, \\ T_2(y) &= 2y^2 - 1, & &\cdots\cdots\cdots\cdots \end{aligned} \tag{9}$$

We note that if $\xi = \cos(\tau/r)$, the sum of the first two terms of (7) is identically zero and the third term is precisely $k \cos \tau$, from the definition of T_r. Thus (7) is satisfied. Physically, this means that with a properly constituted elastic non-linearity there exists *an amplitude* of the forcing term such that a simple subharmonic response will cause the spring force to balance partly the inertia force and partly the external force all the time.

It is easily seen that (7) is not the only differential equation which possesses simple subharmonic periodic solutions. As a matter of fact we have the equation

$$\frac{d^2\xi}{d\tau^2} + r^{-2}\xi + k(-1)^{r/2}T_r(\xi) = k \cos \tau, \qquad \text{(r even integers)} \tag{10}$$

with periodic simple subharmonic solutions

$$\xi = \sin \frac{\tau}{r}, \tag{11}$$

and the equation

$$\frac{d^2\xi}{d\tau^2} + r^{-2}\xi + k(-1)^{(r-1)/2}T_r(\xi) = k \sin \tau \qquad \text{(r odd integers)} \tag{12}$$

with solutions also given by (11).

Transformation with Tchebycheff polynomials. Let us now take a different approach to arrive at Eq (7). Let us return to the linear problem with a forcing term, i.e. Eqs. (5) and (6). If we apply the following transformation to (5) and (6)

$$\xi = T_r(\eta), \tag{13}$$

where η is a function of τ, Eq. (5) becomes

$$\frac{d^2T_r(\eta)}{d\tau^2} + T_r(\eta) + kT_r(\eta) = k \cos \tau \tag{14}$$

and the solution is, from (6) and (13),

$$\eta = \cos \frac{\tau}{r}. \tag{15}$$

Equation (14) may be rewritten as

$$\left[\frac{d^2\xi}{d\tau^2} + \xi\right] - \left[\frac{d^2\eta}{d\tau^2} + \frac{1}{r^2}\eta\right] + \left[\frac{d^2\eta}{d\tau^2} + \frac{1}{r^2}\eta + kT_r(\eta)\right] = k \cos \tau, \tag{16}$$

by adding and subtracting the terms in the second square bracket. The first two square brackets are, however, identically zero with ξ and η given by (6) and (15). It follows that $\eta = \cos(\tau/r)$ must be a solution of

$$\frac{d^2\eta}{d\tau^2} + r^{-2}\eta + kT_r(\eta) = k \cos \tau, \tag{17}$$

which is the same as (7).

Similarly, by putting $\xi = (-1)^{r/2}\, T_r(\eta)$ in (5) we can arrive at (10) and (11). Equation (12) and its solution can be established in the same manner by using a slightly different form of (5).

This approach shows that by a single transformation all the non-linear differential equations possessing simple subharmonics as their solutions can be derived from the linear problem which we understand much better. In other words, the simple subharmonic solutions might be considered merely as different representations of the same linear response solution in various frameworks. The physical significance, if any, of this transformation is not apparent at the present time.

Uniqueness of the solutions. There has been some discussion among interested people concerning the uniqueness of the simple subharmonics as steady-state solutions of (7). Since it is known that the linear response (6) is a unique steady state solution of (5), and since Eq. (7) and its simple subharmonic solutions can be derived from the linear case by a proper transformation, which is one-to-one when continuity conditions are imposed on the solutions, it may be concluded that these simple subharmonic solutions are unique so far as the steady state is concerned.

Bibliography

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2. R. M. Rosenberg, *On the origin of subharmonic vibrations of odd orders*, Proc., 2nd Midwest Conf. on Solid Mechanics, Purdue University, 1955

Correction to my paper
ON THE DAMPED OSCILLATIONS EQUATION WITH VARIABLE COEFFICIENTS

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Equation (13) should have been written

$$\phi(0) > \{\phi(0)u(0)^2 + [u'(0) + p(0)u(0)/2]^2\} \tag{13}$$

that is, the $u'(0)$ should not be squared. Similarly the last equation on page 93 should contain only $\alpha'(0)$ in the brackets.