

Also, $\psi(\lambda)$ is a parabola which takes on its minimum value, zero, at $\lambda = 1$. The function $\varphi(\lambda)$ is always non-negative. Hence $\varphi(\lambda)$ must reach a relative minimum for a value $\lambda = \lambda_m$, where $0 < \lambda_m \leq \infty$. The quantity λ_m may readily be calculated and the corresponding point:

$$R \equiv (\xi_i + \lambda_m du_i)$$

then used as a starting point for the calculation in place of ξ_i . Thus φ must decrease with every iteration. Moreover, the iterative method described above will converge in one step if Eqs. (1) are linear, and it has been found in many calculations that the method converges rapidly if the initial approximation is sufficiently close to the solution.

If the derivatives $\partial f_i / \partial x_i$ are not readily found one may approximate them by difference quotients $\Delta f_i / \Delta x_i$. In extremely non-linear problems where independent and dependent variables become very small or very large in the process of calculation the choice of suitable increments Δx_i may not be obvious. In this case one may take the increments Δx_i to be differences in successive approximations of x_i . It has been found in many calculations that for sufficiently smooth functions approximately maximum numerical accuracy is then obtained for the solution.

The method described here was successful in solving certain extremely non-linear chemical equilibrium equations where independent and dependent variables were unpredictably small or large. In other methods—such as the “steepest descent method”—which were tried the rate of convergence decreased rapidly and the method became impractical as the solution was approached, whereas in this method the rate of convergence increased rapidly. Combinations of the steepest descent method with this method suggest themselves. For example, let

$$x_i = \xi_i + \lambda du_i + \mu dv_i,$$

where μ is a parameter and $dv_i = -\text{grad } \varphi$. The parameters λ and μ are then adjusted so that φ is minimum.

ON CONVERGENT PERTURBATION EXPANSIONS*

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1. Introduction. In this paper, we wish to consider the Sturm-Liouville equation

$$\begin{aligned} u'' + \lambda(f(x) + \epsilon g(x))u &= 0, \\ u(0) &= u(1) = 0, \end{aligned} \tag{1}$$

and the problem of obtaining power series expansions for the first characteristic value and first characteristic function.

Let λ_0 and $u_0(x)$ be respectively the first characteristic value and associated characteristic function of the equation

$$\begin{aligned} u'' + \lambda f(x)u &= 0, \\ u(0) &= u(1) = 0. \end{aligned} \tag{2}$$

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Let $u_0(x)$ be normalized by the condition that $u'(0) = 1$. Then it is well known how to determine perturbation expansions of the form

$$\begin{aligned}\lambda &= \lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \cdots, \\ u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots,\end{aligned}\tag{3}$$

for the smallest characteristic value of (1) and the characteristic function reducing to u_0 when $\epsilon \rightarrow 0$.

In [1], it was shown that in a number of similar situations involving linear operator equations of the form

$$L(u) + \epsilon M(u) = 0,\tag{4}$$

under suitable assumptions it was possible to find a function $h(\epsilon)$ with the property that the perturbation series

$$u = u_0 + u_1 h(\epsilon) + u_2 h(\epsilon)^2 + \cdots\tag{5}$$

converged for all $\epsilon \geq 0$, a case of physical importance. The choice of $h(\epsilon)$ depends upon the *a priori* knowledge of the location of certain singularities of u regarded as a function of ϵ .

In [3], in response to a question posed in [2], the following result was established.

Theorem 1. With reference to the equation, assume that

$$\begin{aligned}\text{(a)} \quad & f(x) \text{ and } g(x) \text{ are continuous for } 0 \leq x \leq 1, \\ \text{(b)} \quad & f(x), g(x) \geq a^2 > 0, \text{ for } 0 \leq x < 1.\end{aligned}\tag{6}$$

Then, Λ , the smallest characteristic root of (1) is an analytic function of ϵ for

$$\operatorname{Re}(\epsilon) \geq b = -\operatorname{Min}_{0 \leq x \leq 1} g(x)/f(x).\tag{7}$$

Furthermore, the only singularity of Λ on the line $b + i\tau$, $-\infty < \tau < \infty$, is at $\tau = 0$.

Using this result, we shall show how to determine $h(\epsilon)$, as above in (5), so as to obtain an expansion for Λ which is convergent for all $\epsilon \geq 0$.

2. The choice of $h(\epsilon)$. Consider the transformation

$$\delta = \epsilon/(k + \epsilon),\tag{1}$$

where k is a real quantity to be specified in a moment.

The unit circle, $|\delta| = 1$, is carried into the line $2\operatorname{Re}(\epsilon) + k = 0$ in the complex ϵ -plane. Hence, if $2\operatorname{Re}(\epsilon) + k > 0$, we have $|\delta| < 1$. Let us then choose k to be a positive quantity in the interval

$$0 < k < -b,\tag{2}$$

where b is defined by (1.7).

Then $\Lambda(\epsilon) = \Lambda[k\delta/(1 - \delta)]$ is analytic for $|\delta| < 1$. Hence, we have an expansion

$$\Lambda(\epsilon) = \Lambda[k\delta/(1 - \delta)] = \sum_{n=0}^{\infty} \lambda_n \delta^n,\tag{3}$$

valid for $|\delta| < 1$, or, finally,

$$\Lambda(\epsilon) = \sum_{n=0}^{\infty} \lambda_n [\epsilon/(k + \epsilon)]^n, \quad (4)$$

valid for $\epsilon \geq 0$, and, actually, for $\operatorname{Re}(\epsilon) > -k$.

3. The expansion for the associated characteristic function. Consider the solution of

$$\begin{aligned} u'' + \Lambda(\epsilon)[f(x) + \epsilon g(x)]u &= 0, \\ u(0) &= 0, \quad u'(0) = 1. \end{aligned} \quad (1)$$

The classical iteration procedure for obtaining u shows that u is an analytic function of $\Lambda(\epsilon)$ and ϵ for $0 \leq x \leq 1$, for all ϵ and $\Lambda(\epsilon)$. Consequently, u is an analytic function of ϵ for $\operatorname{Re}(\epsilon) > b$, uniformly in $0 \leq x \leq 1$, and thus possesses an expansion

$$u = \sum_{n=0}^{\infty} u_n(x) [\epsilon/(k + \epsilon)]^n, \quad (2)$$

valid for $\epsilon \geq 0$.

4. Perturbation procedure. To determine the sequences $\{\lambda_n\}$ and $\{u_n(x)\}$, we employ a standard perturbation technique starting with the equation

$$\begin{aligned} u'' + \lambda[f(x) + k\delta g(x)/(1 - \delta)]u &= 0, \\ u(0) &= u(1) = 0. \end{aligned} \quad (1)$$

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THE NUMERICAL SOLUTION OF AN INFINITE SET OF LINEAR SIMULTANEOUS EQUATIONS*

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The usual method for numerical solution of

$$\sum_{s=1}^{\infty} a_{rs} x_s = b_r, \quad (r = 1, 2, \dots), \quad (1)$$

is to solve the first n equations in n unknowns for some fixed n , and to regard the results as approximations to the x_s . The accuracy of this approximation is invariably doubtful. A slight modification of the Choleski¹ or Crout² method for solving a set of simultaneous linear equations can be used to provide a systematic procedure for solving the first $m \times m$ equations with $m = 1, 2, 3, \dots$ in succession. It is then possible to estimate in various ways the error involved in solving only a finite number of equations. The following discussion should be intelligible from either the Crout or the Choleski point of view.

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²P. D. Crout, Trans. Am. Inst. Elec. Engrs. **60**, 1235-40 (1941).