A MODIFICATION OF PRAGER'S HARDENING RULE*

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1. Introduction. The response of a rigid-work-hardening material can be described by

a) an initial yield condition, specifying the states of stress for which plastic flow first sets in,

b) a flow rule, connecting the plastic strain increment with the stress and the stress increment,

c) a hardening rule, specifying the modification of the yield condition in the course of plastic flow.

It is customary to represent the yield condition as a surface in stress space, convex [1] and initially containing the origin. The current yield conditions for a metal are those of v. Mises [2] and of Tresca [3]. The flow rule generally accepted [4, 1] is also due to v. Mises [5]. It states that the strain increment vector lies in the exterior normal of the yield surface at the stress point. As to the hardening rule, there are mainly two versions in use. The rule of isotropic work-hardening given by Hill and Hodge [6, 7] assumes that the yield surface expands during plastic flow, retaining its shape and situation with respect to the origin. Another rule, developed by Prager [8], assumes that the yield surface is rigid but undergoes a translation in the direction of the strain increment.

The rule of isotropic work-hardening does not account for the Bauschinger effect observed in the materials in question. Prager's hardening rule accounts for this effect. However, as Perrone and Hodge Jr. [9] have shown in special cases and Shield and the author [10] in a general investigation, Prager's hardening rule is not invariant with respect to reductions in dimensions possible in almost any applications. In other words: if the yield surface in 9-space σ_{ik} moves in the direction of the exterior normal at the stress point, the two-dimensional yield locus, e.g., in plane stress σ_x , σ_y does not do so. In certain cases, e.g., if only σ_x and τ_{xy} are different from zero, the Tresca yield locus in the plane σ_x , τ_{xy} even deforms.

It is clear that the physical consistency of Prager's rule is not affected by the phenomena last mentioned. However, they complicate the application of the rule particularly in cases which otherwise would be simple enough to lend themselves to a complete treatment. On the other hand, the investigations of Shield and the author have shown that, under v. Mises' yield condition at least, the yield surface, in all special cases, according to Prager's rule moves in the direction of the radius connecting its center with the stress point. This suggests that a corresponding modification of Prager's rule might simplify certain problems. In the following sections such a modification is formulated, investigated and compared with Prager's rule.

2. The modified hardening rule. Let us consider an element of a rigid-work-hardening solid, referred to an orthogonal coordinate system x_i . The state of stress of this

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element can be represented by a stress point P in a 9-space σ_{ik} with origin O. In this space, the initial yield surface is represented by an equation

$$F(\sigma_{ik}) = k^2 = \text{const.} \tag{2.1}$$

In the following, for simplicity, attention will be confined to initially isotropic materials for which the form of the function F is invariant with respect to a rotation of the stress state. An initially anisotropic material can be treated in an analogous manner.

The hardening rule suggested by Prager assumes that during plastic deformation the yield surface moves in a translation. After a certain amount of plastic flow, it is given by

$$F(\sigma_{ik} - \alpha_{ik}) = k^2, \tag{2.2}$$

where the tensor α_{ik} represents the total translation. Because α_{ik} is not necessarily the isotropic tensor δ_{ik}, where δ_{ik} is the Kronecker delta, *the material becomes anisotropic as a result of the hardening process.* Accordingly, direction is important, and we shall fix the coordinate system x_i with respect to the element, small deformations being assumed.

In the space σ_{ik}, the α_{ik} represent the radius vector of the point C, which initially was at the origin and which in the following will be referred to as the center of the yield surface. Due to the flow rule of v. Mises, the plastic strain increment $d\epsilon_{ik}$, considered as a vector in the same space, lies in the exterior normal of the surface (2.2) at P. Thus, it is represented by

$$d\epsilon_{ik} = \frac{\partial F}{\partial \sigma_{ik}}\, d\lambda, \qquad d\lambda > 0. \tag{2.3}$$

The definition of a Prager-hardening material is completed by assuming that the surface (2.2) moves in the direction of $d\epsilon_{ik}$; more explicitly

$$d\alpha_{ik} = c\, d\epsilon_{ik}, \tag{2.4}$$

where c is a constant characterizing the material.

Instead of (2.4), let us assume

$$d\alpha_{ik} = (\sigma_{ik} - \alpha_{ik})\, d\mu, \qquad d\mu > 0, \tag{2.5}$$

i.e. that the yield surface still moves in a translation, but in the direction of the vector CP connecting the center of the yield surface with the stress point (Fig. 1). This rule

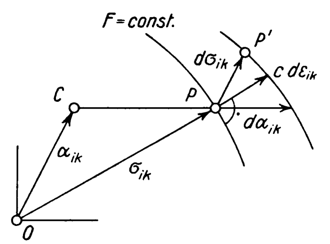

FIG. 1. Hardening rule and flow rule for a linear work-hardening solid.

is a modification of Prager's law, physically acceptable since both sides of (2.5) are tensors of the second order.

The scalar $d\mu$ in (2.5) is determined by the condition that P remains on the yield surface in plastic flow. If in 9-space the summation convention is adopted, this condition is

$$(d\sigma_{ik} - d\alpha_{ik}) \frac{\partial F}{\partial \sigma_{ik}} = 0, \tag{2.6}$$

and from (2.5) follows at once

$$d\mu = \frac{(\partial F/\partial \sigma_{ij})\, d\sigma_{ij}}{(\sigma_{kl} - \alpha_{kl})\, \partial F/\partial \sigma_{kl}}. \tag{2.7}$$

Since (2.4) has been replaced by (2.5), the vector OC no longer represents the total strain. This is a serious drawback of our modification of Prager's rule. On the other hand, we gain a substantial advantage: $d\lambda$ in (2.3), i.e. the magnitude of the strain increment remains free and can still be established as a suitable function of the stress, the stress increment and the stress history. In other words: *it is possible to adjust the modified hardening rule* (2.5) *to any kind of hardening law in simple tension and compression.*

The simplest way to dispose of $d\lambda$, i.e. to complete the flow rule, is to assume that the vector $c d\epsilon_{ik}$ is the projection of $d\sigma_{ik}$ (and thus of $d\alpha_{ik}$) on the exterior normal of the yield surface. This corresponds to the procedure familiar from Prager's rule, and it will turn out later on that on this basis the results of either rule coincide in many cases. If $d\lambda$ is fixed in this way, the total strain is represented by the sum of the infinitesimal translations of the yield surface in the direction of its exterior normal at the stress point. Since

$$(d\sigma_{ik} - c\, d\epsilon_{ik}) \frac{\partial F}{\partial \sigma_{ik}} = 0, \tag{2.8}$$

we obtain from (2.3)

$$d\lambda = \frac{1}{c} \frac{(\partial F/\partial \sigma_{ij})\, d\sigma_{ij}}{(\partial F/\partial \sigma_{kl})\, (\partial F/\partial \sigma_{kl})}. \tag{2.9}$$

By means of the hardening rule (2.5), (2.7) and the flow rule (2.3), (2.9) the material is completely defined. It is easy to see that these rules are a generalization to complex states of stress of a linear hardening law in tension and compression, Fig. 2, which exhibits a Bauschinger effect. Moreover, by assuming that c is not a constant, but a

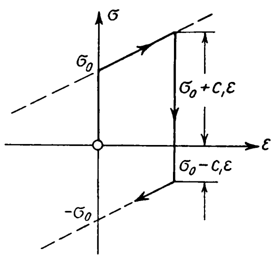

FIG. 2. Linear work-hardening in simple tension and compression.

suitable function of the stress history, e.g. of the distance OC or of the dissipation work, it is possible to adapt our rules to any material with a given non-linear hardening law in simple tension and compression.

For most of the following considerations it will not be necessary to restrict ourselves to linear work-hardening materials, i.e. to constant values of c.

3. Some general properties. In an initially isotropic solid the yield function takes the form

$$F(\sigma_{ik}) = G[I_1(\sigma_{ik}), I_2(\sigma_{ik}), I_3(\sigma_{ik})], \tag{3.1}$$

where

$$I_1 = \sigma_{ii}, \qquad I_2 = \tfrac{1}{2}\sigma_{ij}\sigma_{ji}, \qquad I_3 = \tfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} \tag{3.2}$$

are the invariants of the stress tensor. If the initial yield is independent of the mean normal stress,

$$F(\sigma_{ik} + \beta\delta_{ik}) = F(\sigma_{ik}), \tag{3.3}$$

where β is an arbitrary scalar. When plastic flow has set in, the yield function becomes, on account of (2.2) and (3.1),

$$F(\sigma_{ik} - \alpha_{ik}) = G[I_1(\sigma_{ik} - \alpha_{ik}), I_2(\sigma_{ik} - \alpha_{ik}), I_3(\sigma_{ik} - \alpha_{ik})]. \tag{3.4}$$

From (3.3) follows that the values of (3.4) remain unchanged when σ_{ik} is replaced by $\sigma_{ik} + \beta\delta_{ik}$: *if yield is initially independent of the mean normal stress, it remains so.*

On account of (3.4), the flow rule (2.3) becomes

$$d\epsilon_{ik} = \left(\frac{\partial G}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ik}} + \frac{\partial G}{\partial I_2}\frac{\partial I_2}{\partial \sigma_{ik}} + \frac{\partial G}{\partial I_3}\frac{\partial I_3}{\partial \sigma_{ik}}\right) d\lambda. \tag{3.5}$$

Carrying out the differentiation of the invariants, we obtain

$$d\epsilon_{ik} = \left[\frac{\partial G}{\partial I_1}\delta_{ik} + \frac{\partial G}{\partial I_2}(\sigma_{ik} - \alpha_{ik}) + \frac{\partial G}{\partial I_3}(\sigma_{ij} - \alpha_{ij})(\sigma_{jk} - \alpha_{jk})\right] d\lambda. \tag{3.6}$$

Let us assume now that the physical coordinate system initially coincides with the principal axes of stress. Then we have first

$$\sigma_{ik} = 0, \qquad (i \neq k) \quad \text{and} \quad \alpha_{ik} = 0. \tag{3.7}$$

From (2.5) and from (3.6) follows

$$d\alpha_{ik} = 0, \qquad (i \neq k) \quad \text{and} \quad d\epsilon_{ik} = 0, \qquad (i \neq k). \tag{3.8}$$

Since the material is isotropic at the outset, this means that the strain increment tensor $d\epsilon_{ik}$ and the tensor $d\alpha_{ik}$ are coaxial with the stress tensor σ_{ik}. The relations (3.8) remain valid if the second assumption (3.7) is replaced by the weaker assumption

$$\alpha_{ik} = 0, \qquad (i \neq k). \tag{3.9}$$

It follows that, *if the principal axes of stress remain fixed in the element from the start, the strain increment tensor and thus the strain tensor remain coaxial with the stress tensor.*

If the principal axes of stress rotate, (3.8) holds only in a first step, provided the principal system of stress is used as the physical coordinate system. If (3.8) shall hold in a second step, the coordinate system must be rotated between the first step and the second one. This rotation, however, violates (3.9): *due to the anisotropy caused by strain hardening, the strain increment tensor is in general not coaxial with the stress tensor.*

Many problems of practical importance can be treated in a stress space of less than 9 dimensions. In certain cases, e.g., a 3-space defined by the principal stresses is useful. From our last result follows, however, that this 3-space is inadequate where the principal axes of stress are not fixed in the element.

The properties obtained in this section apply without exception also for a material obeying Prager's rule [10].

4. Treatment in subspaces. On account of the symmetry of the stress and strain tensors, the problem may as well be treated in 6-space. It is convenient here and particularly for the subsequent specializations to denote the physical coordinates by x, y, z, the stresses by $\sigma_x , \cdots , \tau_{yz} , \cdots$, and the strains by $\epsilon_x , \cdots , \epsilon_{yz} , \cdots$, where the dots indicate cyclic permutations.

In the new notations the yield condition (2.2) reads

$$F(\sigma_x - \alpha_x , \cdots , \tau_{yz} - \alpha_{yz} , \cdots , \tau_{zy} - \alpha_{zy} , \cdots) = k^2, \tag{4.1}$$

where τ_{yz} , τ_{zy} , $\cdots$ have to be considered as independent variables. The flow rule (2.3) becomes

$$d\epsilon_x = \frac{\partial F}{\partial \sigma_x} d\lambda, \cdots , d\epsilon_{yz} = \frac{\partial F}{\partial \tau_{yz}} d\lambda, \cdots , d\epsilon_{zy} = \frac{\partial F}{\partial \tau_{zy}} d\lambda, \cdots , \tag{4.2}$$

and the hardening rule (2.5) takes the form

$$\left.\begin{aligned} d\alpha_x = (\sigma_x - \alpha_x)\, d\mu, \cdots , d\alpha_{yz} &= (\tau_{yz} - \alpha_{yz})\, d\mu, \cdots , \\ d\alpha_{zy} &= (\tau_{zy} - \alpha_{zy})\, d\mu, \cdots . \end{aligned}\right\}. \tag{4.3}$$

Treatment in 6-space, however, requires the elimination of the stress components τ_{zy} $(= \tau_{yz})$, $\cdots$, of the strain components ϵ_{zy} $(= \epsilon_{yz})$, $\cdots$, and of the displacements α_{zy} , $\cdots$, which, on account of (4.3), are equal to the displacements α_{yz} , $\cdots$. On account of the symmetry of the stress tensor

$$F(\sigma_x , \cdots , \tau_{yz} , \cdots , \tau_{zy} , \cdots) = f(\sigma_x , \cdots , \tau_{yz} , \cdots). \tag{4.4}$$

Thus, the yield surface in 6-space is given by

$$\begin{aligned} &f(\sigma_x - \alpha_x , \cdots , \tau_{yz} - \alpha_{yz} , \cdots) \\ &\qquad = F[\sigma_x - \alpha_x , \cdots , \tau_{yz} - \alpha_{yz} , \cdots , \tau_{zy} - \alpha_{zy} , \cdots] = k^2. \end{aligned} \tag{4.5}$$

From (4.2) and (4.5) we obtain

$$d\epsilon_x = \frac{\partial f}{\partial \sigma_x} d\lambda, \cdots , \qquad d\gamma_{yz} = 2\, d\epsilon_{yz} = \frac{\partial f}{\partial \tau_{yz}} d\lambda, \cdots . \tag{4.6}$$

This is the well known result that *the flow rule of v. Mises remains valid in 6-space, if the state of strain is represented by the engineering components* $\epsilon_x , \cdots , \gamma_{yz} , \cdots$.

Further, from (4.3) follows

$$d\alpha_x = (\sigma_x - \alpha_x)\, d\mu, \cdots , \qquad d\alpha_{yz} = (\tau_{yz} - \alpha_{yz})\, d\mu, \cdots . \tag{4.7}$$

Thus, *the hardening rule* (2.5) *applies without modification also in 6-space.*

In many practically important cases, some of the stress components are identically zero. Starting once more in 9-space, let us denote the stress components present by σ'_{ik} , the zero ones by σ''_{ik} . The initial yield condition is then

$$F(\sigma'_{ik} , \sigma''_{ik} = 0) = H(\sigma'_{ik}) = k^2. \tag{4.8}$$

If we are not interested in the strains ϵ''_{ik} corresponding to the zero stresses σ''_{ik} , we may treat the problem in a subspace σ'_{ik} . Here $H(\sigma'_{ik})$ defines a new yield surface.

On account of the hardening rule (2.5)

$$\alpha''_{ik} = 0. \tag{4.9}$$

Thus, the yield surface, after plastic flow has set in, is given by

$$F(\sigma'_{ik} - \alpha'_{ik}, \sigma''_{ik} - \alpha''_{ik} = 0) = H(\sigma'_{ik} - \alpha'_{ik}) = k^2. \tag{4.10}$$

It follows that *also in the subspace* σ'_{ik} *the yield surface moves in a translation.*

On account of (4.10)

$$d\epsilon'_{ik} = \frac{\partial F}{\partial \sigma'_{ik}} d\lambda = \frac{\partial H}{\partial \sigma'_{ik}} d\lambda. \tag{4.11}$$

Thus, *the flow rule remains valid in any subspace.* Of course, it supplies only the strain components ϵ'_{ik} defined in this subspace, although the ϵ''_{ik}, too, may be different from zero.

Finally, from (2.5) follows

$$d\alpha'_{ik} = (\sigma'_{ik} - \alpha'_{ik})\, d\mu, \tag{4.12}$$

i.e., *also the hardening rule* (2.5) *remains valid in any subspace.* This is another advantage in comparison with Prager's rule which in most subspaces applies only in a modified form [10].

In the next sections, we shall discuss a few special states of stress particularly important in applications. We shall restrict ourselves to materials where yield is independent of the mean normal stress. Here, as in (3.3),

$$f(\sigma_x - \alpha_x + \beta, \cdots, \tau_{yz} - \alpha_{yz}, \cdots) = f(\sigma_x - \alpha_x, \cdots, \tau_{yz} - \alpha_{yz}, \cdots). \tag{4.13}$$

5. Plane strain. Here,

$$\tau_{yz} = \tau_{zx} = 0, \qquad \epsilon_z = 0, \tag{5.1}$$

by definition. From (4.9) follows $\alpha_{yz} = \alpha_{zx} = 0$. Thus, the yield function has the form

$$g(\sigma_x - \alpha_x, \sigma_y - \alpha_y, \sigma_z - \alpha_z, \tau_{xy} - \alpha_{xy}). \tag{5.2}$$

On account of (5.1) and the flow rule (4.6), σ_z must be absent from (5.2). Using (4.13), we finally obtain the yield condition

$$h(\sigma_x - \alpha'_x, \sigma_y - \alpha'_y, \tau_{xy} - \alpha_{xy}) = k^2, \tag{5.3}$$

where

$$\alpha'_x = \alpha_x - \alpha_z \quad \text{and} \quad \alpha'_y = \alpha_y - \alpha_z .$$

If the material obeys v. Mises' yield condition, we have initially

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = \frac{4}{3}\sigma_0^2, \tag{5.4}$$

where σ_0 is the yield limit in simple tension or compression. The yield surface therefore is the elliptic cylinder

$$[(\sigma_x - \alpha'_x) - (\sigma_y - \alpha'_y)]^2 + 4(\tau_{xy} - \alpha_{xy})^2 = \frac{4}{3}\sigma_0^2 \tag{5.5}$$

with its axis parallel to the line bisecting the angle of the first quadrant and with semi-axes $(2/3)^{1/2}\sigma_0$, $(1/3)^{1/2}\sigma_0$.

If Tresca's yield condition holds, we have, instead of (5.4),

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = \sigma_0^2 . \tag{5.6}$$

The yield surface is again an elliptic cylinder

$$[(\sigma_x - \alpha_x') - (\sigma_y - \alpha_y')]^2 + 4(\tau_{xy} - \alpha_{xy})^2 = \sigma_0^2 \tag{5.7}$$

with its axis in the same direction, but with semiaxes $\sigma_0/2^{1/2}$, $\sigma_0/2$.

Subcase a: If $\tau_{xy} = 0$, (4.9) yields $\alpha_{xy} = 0$. The problem can be treated in a plane σ_x, σ_y, and the yield locus is the strip (Fig. 3) obtained by bisecting the cylinder (5.5) or (5.7) parallel to the plane σ_x, σ_y.

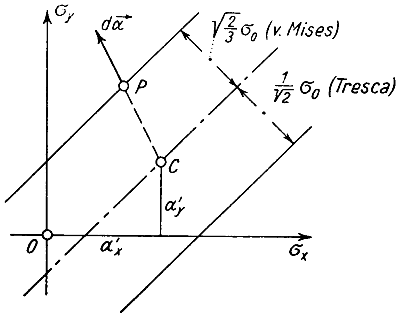

FIG. 3. Yield locus in plane strain with $\tau_{xy} = 0$.

Subcase b: If $\sigma_y = 0$, (4.13) can be used once more, and the problem can be treated in a plane σ_x, τ_{xy}. The yield locus is the ellipse (Fig. 4) obtained by bisecting the cylinder (5.5) or (5.7) parallel to the plane σ_x, τ_{xy}. Its semiaxes are $2\sigma_0/3^{1/2}$, $\sigma_0/3^{1/2}$ for v. Mises' yield condition and σ_0, $\sigma_0/2$ for the condition of Tresca.

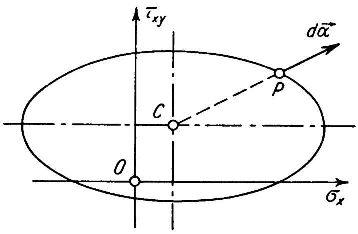

FIG. 4. Yield locus in plane strain with $\sigma_y = 0$.

Since the length of the cylinder (5.5) or (5.7) is infinite, it does not matter whether its translation is given by the vector $d\alpha$ in the direction CP or by its projection on the exterior normal. It follows that for a linear work-hardening law [c in (2.9) constant] the results obtained here coincide with those supplied by Prager's hardening rule.

6. Plane stress. Here, by definition,

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0. \tag{6.1}$$

From (4.9) follows $\alpha_z = \alpha_{yz} = \alpha_{zx} = 0$. Thus, the yield surface has the form

$$g(\sigma_x - \alpha_x , \sigma_y - \alpha_y , \tau_{xy} - \alpha_{xy}) = k^2. \tag{6.2}$$

If the material obeys v. Mises' yield condition, we have initially

$$\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 = \sigma_0^2 \tag{6.3}$$

and thus

$$(\sigma_x - \alpha_x)^2 + (\sigma_y - \alpha_y)^2 - (\sigma_x - \alpha_x)(\sigma_y - \alpha_y) + 3(\tau_{xy} - \alpha_{xy})^2 = \sigma_0^2 . \tag{6.4}$$

The yield surface is an ellipsoid with semiaxis $2^{1/2}\sigma_0$ in the direction of the line bisecting the first quadrant in the plane σ_x , σ_y , with semiaxis $(2/3)^{1/2}\sigma_0$ in the direction of the line bisecting the second quadrant and $\sigma_0/3^{1/2}$ in the direction of the axis τ_{xy} .

If Tresca's condition holds, we have initially the yield limits

$$[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]^{1/2} = \sigma_0 \tag{6.5}$$

and

$$\tfrac{1}{2} \mid \sigma_x + \sigma_y \pm [(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]^{1/2} \mid = \sigma_0 . \tag{6.6}$$

Thus, the yield surface is an elliptic cylinder with the same orientation as v. Mises ellipsoid, closed by two elliptic cones.

Subcase a: If $\tau_{xy} = 0$, (4.9) supplies $\alpha_{xy} = 0$, and the problem can be treated in a plane σ_x , σ_y . If v. Mises' yield condition (6.4) applies, the yield locus is the ellipse of Fig. 5 with the equation

$$(\sigma_x - \alpha_x)^2 + (\sigma_y - \alpha_y)^2 - (\sigma_x - \alpha_x)(\sigma_y - \alpha_y) = \sigma_0^2 . \tag{6.7}$$

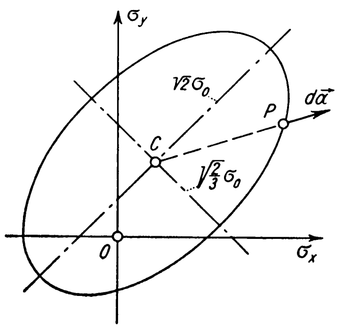

FIG. 5. v. Mises yield locus in plane stress with $\tau_{xy} = 0$.

If the material obeys Tresca's yield condition (6.5), (6.6), the yield locus is the hexagon of Fig. 6.

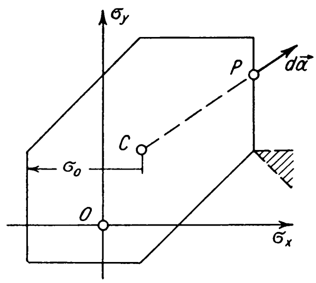

FIG. 6. Tresca yield locus in plane stress with $\tau_{xy} = 0$.

Subcase b: If $\sigma_y = 0$, (4.9) yields $\alpha_y = 0$, and the problem can be treated in a plane σ_x, τ_{xy}. The yield locus is the ellipse of Fig. 4 with semiaxes σ_0, $\sigma_0/3^{1/2}$ in v. Mises' case and σ_0, $\sigma_0/2$ in the case of Tresca.

If P lies in a corner or vertex of the yield surface, the strain increment exhibits the same indeterminacy in direction as in a perfectly plastic material. In the lower right corner of the Tresca hexagon of Fig. 6, e.g., the vector $d\boldsymbol{\varepsilon}$ lies anywhere in the shaded region defined by the normals of the adjacent sides. However, P remains only in the corner or vertex if the yield surface moves with P, i.e. if the stress increment $d\boldsymbol{\sigma}$ has the direction CP.

7. Another special case. In certain cases, e.g. in a cylinder subjected to torsion and and tension, we have

$$\sigma_x = \sigma_y = \tau_{xy} = 0. \tag{7.1}$$

From (4.9) follows $\alpha_x = \alpha_y = \alpha_{xy} = 0$. Thus, the yield function has the form

$$g(\sigma_z - \alpha_z, \tau_{yz} - \alpha_{yz}, \tau_{zx} - \alpha_{zx}) = k^2. \tag{7.2}$$

If the material obeys v. Mises' yield condition, we have initially

$$\sigma_z^2 + 3(\tau_{yz}^2 + \tau_{zx}^2) = \sigma_0^2 \tag{7.3}$$

and thus

$$(\sigma_z - \alpha_z)^2 + 3(\tau_{yz} - \alpha_{yz})^2 + 3(\tau_{zx} - \alpha_{zx})^2 = \sigma_0^2 . \tag{7.4}$$

The yield surface is an ellipsoid of revolution with semiaxes σ_0, $\sigma_0/3^{1/2}$.

If Tresca's yield condition applies, the principal stresses are initially

$$\sigma_1 = 0, \qquad \sigma_{2,3} = \tfrac{1}{2}[\sigma_z \pm \{\sigma_z^2 + 4(\tau_{yz}^2 + \tau_{zx}^2)\}^{1/2}]. \tag{7.5}$$

The maximum shear stress is $(\sigma_2 - \sigma_3)/2$; hence, we have, instead of (7.4),

$$(\sigma_z - \alpha_z)^2 + 4(\tau_{yz} - \alpha_{yz})^2 + 4(\tau_{zx} - \alpha_{zx})^2 = \sigma_0^2 , \tag{7.6}$$

i.e. an ellipsoid of revolution with semiaxes σ_0, $\sigma_0/2$.

Subcase a: If $\sigma_z = 0$, (4.9) supplies $\alpha_z = 0$. The problem can be treated in a plane τ_{yz}, τ_{zx}. The yield locus is a circle of radius $\sigma_0/3^{1/2}$ in v. Mises' case and $\sigma_0/2$ in the case of Tresca.

Subcase b: If $\tau_{yz} = 0$, (4.9) yields $\alpha_{yz} = 0$, and the problem can be treated in a plane σ_z, τ_{zx}. The yield locus is an ellipse as in Fig. 4 with semiaxes σ_0, $\sigma_0/3^{1/2}$ in v. Mises' case and σ_0, $\sigma_0/2$ in the case of Tresca. It is clear that this result, apart from the difference in notation, is the one of Subcase *b* in Sec. 6.

8. Discussion. By suitable linear transformations of the stresses (and of the α_{ik}) the yield surfaces and yield loci obtained in Secs. 5 through 7 could be simplified. Ellipsoids can be transformed into spheres, ellipses into circles and so on. The hardening rule (2.5) is not affected by such a process. However, if the flow rule (2.3) is to apply in the new variables, it is necessary to transform the strains [10, 11] simultaneously.

At the present stage of research, it seems hopeless to expect a decision by experiment between the hardening rules confronted here. Comparisons must be based, therefore, on purely theoretical reasoning.

It has been pointed out in Sec. 5 that in plane strain the results obtained for a linear work-hardening material by the hardening rule (2.5) are the same as those supplied by

Prager's rule. A comparison with [10] shows that the situation is the same in plane stress and in the stress state of Sec. 7, provided that the material obeys v. Mises' yield condition. So far it does not matter which hardening rule is used. If, however, a material obeying Tresca's yield condition is subjected to plane stress or to the state of stress of Sec. 7, the response depends on the hardening rule. Under Prager's rule, the yield surface deforms in most cases. Under the rule (2.5), such deformations do not occur, since, according to a statement in Sec. 4, the hardening rule (2.5) applies without modification in any subspace. This is a definite advantage of the rule (2.5).

Another point has been raised in Sec. 2: the possibility of adjusting the hardening rule (2.5) to arbitrary non-linear hardening laws in simple tension and compression. This advantage, however, is obtained at the expense of the geometrical interpretation of the total strain. If one is prepared to renounce this possibility, one clearly is in a position also to adjust Prager's rule to non-linear laws. This point, therefore, is hardly of importance.

A serious drawback of the hardening rule (2.5) lies in the indeterminacy of the strain increment in a corner or vertex of the yield surface. This problem arises already if the Tresca yield condition is applied to a perfectly plastic solid.

Let us assume that a perfectly plastic cylinder with axis x is subjected to simple tension. In plastic flow, the stress point has the position P in Fig. 7. The exterior normal

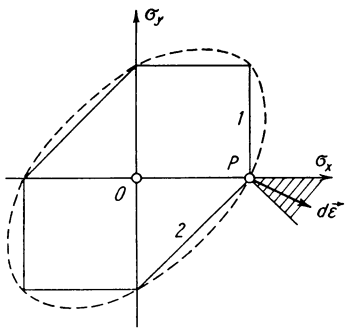

FIG. 7. v. Mises and Tresca yield in simple tension.

at P of v. Mises' ellipse has two components with the ratio $2: (-1)$. Hence, the contraction of the cylinder in the direction y is half the elongation in the direction x and thus the same as the contraction in the direction z, since the volume change is zero. In other words: if v. Mises' yield condition applies, the deformation of the cylinder exhibits the same rotational symmetry with respect to the axis x as the state of stress. However, if the material obeys Tresca's yield condition, the direction of $d\varepsilon$ is free within the shaded angle formed by the normals in P of the sides 1 and 2 of the Tresca hexagon. In view of the incompressibility of the material this means that under Tresca's yield condition the deformation may be arbitrarily unsymmetric. There is no reason why the cylinder should not contract, e.g., in the direction y alone. Experimental evidence seems to decide against Tresca's yield condition and its associated flow rule. It is clear, however, that in experiments an exactly linear state of stress cannot be realized. In any event, if we accept the Tresca yield condition together with the flow rule of v. Mises, we have to admit the possibility of an arbitrarily asymmetric deformation in simple tension and compression.

The difficulty considered here is not present in a material work-hardening in accordance with Prager's rule. Here, the strain increment in a corner of the Tresca hexagon is uniquely determined [10] by the stress increment, and the ratio $d\epsilon_x/d\epsilon_y$ is 2: (-1) in simple tension, no matter which yield condition is used. However, a material hardening according to the rule (2.5) exhibits the same indeterminacy in $d\varepsilon$ as a perfectly plastic material. In this respect the rule (2.5) is not quite satisfactory.

We intend to compare the two rules in various examples. So far, it seems that the rule (2.5) is inferior to Prager's hardening rule from a physical point of view, but easier to handle in certain classes of applications.

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