THERMAL INSTABILITY OF VISCOUS FLUIDS*

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I. Introduction

The stability of a viscous fluid in an insulated vertical tube or between insulated vertical planes when a negative temperature gradient is maintained in the upward direction depends on the magnitude of this gradient, the gravity, the geometry of the solid boundary, the properties of the fluid, and the wave length of the disturbance. The purpose of this paper is to present the relationships between these variables for neutral stability, and the results concerning the effect of rotation on stability.

The problem of stability of two superposed fluids in a cylindrical tube with surface tension at the common interface was solved by Maxwell in [1], without consideration of viscosity or wall effects. The stability of a layer of viscous fluid between two infinite horizontal planes when heated from below was investigated by Rayleigh [2], Jeffrey [3], Low [4], and Pellew and Southwell [5]. The problem under consideration resembles Maxwell's in geometry and Rayleigh-Jeffrey's in the means of producing instability as well as in the nature of the physical process through which stability is maintained under certain conditions. This problem has already been considered by Hales [11], Taylor [12], and Ostrach [13], but only incompletely. The contributions of these authors will be referred to at the appropriate places in this paper.

As will be seen, the curves for neutral stability differ from those ordinarily obtained in investigations of hydrodynamic stability in that the critical Rayleigh number occurs at zero wave number, and hence that these curves have no lower branch.

The "principle of exchange of stabilities", which is assumed in many investigations of hydrodynamic stability but proved only in one instance (Pellew and Southwell, [5]), has been shown to be valid without general rotation. With the presence of general rotation, this principle is valid under certain restrictions. The investigation of the effect of rotation lends some support to the belief that rotation has no effect on the onset of instability.

II. Stability of Fluid Between Plane Walls

1. Formulation of the problem. If a layer of heat-conducting viscous fluid between two vertical planes is heated from below, free convection will occur only if the (negative) temperature gradient in the vertical direction is sufficiently great in magnitude. The stability of such a fluid layer is discussed in this section.

Using Cartesian coordinates (x_1, x_2, x_3), with x_3 measured in the upward vertical direction, one can write the equations of motion as

$$\rho\left(\frac{\partial u_i}{\partial \tau} + u_j \frac{\partial u_i}{\partial x_j}\right) = (0, 0, -g)\rho - \frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\rho \frac{\partial u_i}{\partial x_j}\right), \qquad (i = 1, 2, 3) \tag{1}$$

in which the summation convention has been used, ρ is the density, τ is the time, g is the gravitational acceleration, p is the pressure, and ν is the kinematic viscosity, which

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is assumed to be constant. The velocity component in the ith direction is denoted by u_i, and the three quantities in the parenthesis on the right-hand side of Eq. (1) are associated with indices 1, 2, and 3, respectively. The equation of continuity is

$$\frac{\partial \rho}{\partial \tau} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0, \tag{2}$$

and the diffusion equation is

$$\rho\left(\frac{\partial T}{\partial \tau} + u_i \frac{\partial T}{\partial x_i}\right) = \kappa \frac{\partial}{\partial x_i}\left(\rho \frac{\partial T}{\partial x_i}\right), \tag{3}$$

in which T is the absolute temperature and κ the thermal diffusivity, assumed to be constant. For moderate temperature differences the equation of state can be approximated by

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{4}$$

in which T_0 and ρ_0 are the temperature and density, respectively, of the fluid at a point chosen to be the origin, and α is the thermal expansivity.

If convection is present, the temperature, pressure, and density will differ from their mean values. One can write

$$\begin{aligned} T &= T_m + T', \\ p &= p_m + p', \\ \rho &= \rho_m + \rho', \end{aligned} \tag{5}$$

in which the subscript m indicates primary quantities and the primes indicate the perturbation quantities. For the primary temperature distribution

$$T_m = T_0 + \beta x_3 , \tag{6}$$

one has, from Eq. (4),

$$\rho_m = \rho_0(1 - \alpha\beta x_3), \tag{7}$$

so that the hydrostatic pressure is given by

$$\frac{\partial p}{\partial x_3} = -g\rho_0(1 - \alpha\beta x_3). \tag{8}$$

The thermal expansivity for water under normal conditions is of the order of 0.0001/°F, that for air is of the order of 0.002/°F. Thus if the maximum temperature difference is not excessive, the actual change in density is small. It will be assumed here once and for all that the only effect of density change is on the body force per unit volume due to gravity—the effect on the inertia or specific heat capacity being neglected. Although the change in specific weight is small because α is small, it must in no circumstances be neglected, because this change is the motivating force of any convection, and the sole cause of instability.

The requirement of small change in density imposed a limitation to the magnitude of the maximum value of βx_3 or of x_3. If later one does not hesitate to speak of "zero wave number," which corresponds to infinite wave length in the x_3-direction, it will be with the understanding that that term is only a convenient expression for "long wave lengths." As will be seen, the rate of change of the Rayleigh number with wave number at neutral stability is small when the latter is small, so that the result for "zero wave number" applies rather accurately for long wave lengths.

Under the assumptions made on the effects of the density change, neglecting quadratic terms of the perturbation quantities, and remembering that the undisturbed state is one of dynamic and thermal equilibrium, one can write Eqs. (1) to (3) as

$$\frac{\partial u_i}{\partial \tau} = (0, 0, g\alpha T') - \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu \Delta u_i \, , \tag{9}$$

$$\frac{\partial \rho'}{\partial \tau} - \alpha\beta\rho_0 u_3 + \rho_0 \frac{\partial u_i}{\partial x_i} = 0, \tag{10}$$

$$\left(\frac{\partial}{\partial \tau} - \kappa\Delta\right)T' = -\beta u_3 \, , \tag{11}$$

in which Δ is the Laplacian operator in Cartesian coordinates. By virtue of Eq. (4), Eq. (10) can be written as

$$-\alpha\left(\frac{\partial T'}{\partial \tau} + \beta u_3\right) + \frac{\partial u_i}{\partial x_i} = 0,$$

which, because of the smallness of the thermal expansivity, becomes

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{12}$$

In other words, the effect of change of density on continuity can be neglected. Eliminating p' from the first and third equations contained in Eq. (9), one has,

$$\frac{\partial}{\partial \tau}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = -g\alpha\frac{\partial T'}{\partial x_1} + \nu\Delta\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right). \tag{13}$$

For further discussion u_2 will be assumed to be zero. This does not mean that the motion under consideration is necessarily two dimensional in the usual sense of the word, for u_1 and u_3 may still depend on x_2. If u_2 is zero, however, the equation of continuity permits the use of Lagrange's stream function:

$$u_1 = -\frac{\partial \psi}{\partial x_3}, \qquad u_3 = \frac{\partial \psi}{\partial x_1}. \tag{14}$$

Thus Eqs. (11) and (13) can be written as

$$\left(\frac{\partial}{\partial \tau} - \kappa\Delta\right)T' = -\beta\frac{\partial \psi}{\partial x_1}, \tag{15}$$

$$\left(\frac{\partial}{\partial \tau} - \nu\Delta\right)\Delta\psi = g\alpha\frac{\partial T'}{\partial x_1}. \tag{16}$$

If the origin of the coordinates is taken midway between the plates and the half spacing is denoted by d, the boundary conditions are

$$\psi = 0, \qquad \frac{\partial \psi}{\partial x_1} = 0, \quad \text{and} \quad \frac{\partial T'}{\partial x_1} = 0 \quad \text{at} \quad x_1 = \pm d. \tag{17}$$

For convective motion independent of the coordinate x_2 and periodic in the x_3-direction, one tries solutions of the form

$$\psi = \kappa f(x) \cos az\, e^{\sigma t}, \tag{18}$$

$$T' = \beta\, d\theta(x) \cos az\, e^{\sigma t}, \tag{19}$$

in which

$$(x, y, z) = \left(\frac{x_1}{d}, \frac{x_2}{d}, \frac{x_3}{d}\right), \qquad t = \frac{\kappa\tau}{d^2},$$

a is the dimensionless wave number, and σ (equal to $\sigma_r + i\sigma_i$) is the complex amplification factor. Equations (15) and (16) now become

$$[\sigma - (D^2 - a^2)]\theta = -Df, \tag{20}$$

$$[\sigma - Pr(D^2 - a^2)](D^2 - a^2)f = -R\, Pr\, D\theta, \tag{21}$$

in which D denotes differentiation with respect to x, Pr is the Prandtl number ν/κ, and

$$R = -\frac{g\alpha\beta\, d^4}{\kappa\nu} \tag{22}$$

is the Rayleigh number. The boundary conditions are

$$f = 0, \qquad Df = 0, \quad \text{and} \quad D\theta = 0 \quad \text{at} \quad x = \pm 1. \tag{23}$$

If u_2 is zero but the motion is assumed to be periodic both in x_2 and x_3, one tries solutions of the type

$$\psi = \kappa f(x) \cos \quad by \quad \cos az\, e^{\sigma t}, \tag{24}$$

$$T' = \beta\, d\theta(x) \cos \quad by \quad \cos az\, e^{\sigma t}. \tag{25}$$

Equations (15) and (16) now have the form

$$[\sigma - (D^2 - b^2 - a^2)]\theta = -Df, \tag{26}$$

$$[\sigma - Pr(D^2 - b^2 - a^2)](D^2 - b^2 - a^2)f = -R\, Pr\, D\theta. \tag{27}$$

The boundary conditions are the same as those for the strictly two-dimensional case, but the differential equations now correspond to truly cellular convection of a viscous fluid. Since Eqs. (26) and (27) would be identical to Eqs. (20) and (21) if $a^2 + b^2$ were replaced by a^2, the stability or instability against cellular convection can be predicted from that against the formulation of vortex tubes. This result is similar to that of Squire [6].

2. Principle of exchange of stabilities. In investigations of hydrodynamic stability other than that of the Tollmien-Schlichting type, it has often been assumed that the imaginary part of the factor σ is equal to zero as well as the real part at neutral stability. This is the so-called principle of exchange of stabilities. Only in the case of thermal instability of a viscous fluid between horizontal plates has it been rigorously proved (Pellew and Southwell, [5]). This principle will now be proved for the problem formulated in the last section, but without any assumption concerning boundary geometry.

If the velocity and the pressure are assumed to be periodic vertically, and if

$$u_i = U_i e^{\sigma\tau}, \qquad p' = P e^{\sigma\tau}, \qquad T' = \theta e^{\sigma\tau}, \tag{28}$$

the proof can be achieved by the use of Green's theorems:

$$\iint m_i F_i \, dS = \iiint_V \frac{\partial F_i}{\partial x_i} \, dV, \tag{29}$$

$$\iint_S F \frac{\partial G}{\partial n} \, dS = \iiint_V (\text{grad } F)\cdot(\text{grad } G) \, dV + \iiint_V F \Delta G \, dV, \tag{30}$$

in which m_i are the direction cosines of the outwardly drawn normal to the surface S enclosing a cellular space, and n is the distance along this normal. Multiplying Eq. (9) by u_i^*, summing over i, and utilizing Eq. (12), one has

$$\sigma J_0 = g\alpha H - \frac{1}{\rho_0} \iiint_V \frac{\partial(PU_i^*)}{\partial x_i} \, dV + \nu \iiint_V U_i^* \Delta U_i \, dV, \tag{31}$$

in which

$$J_0 = \iiint_V U_i U_i^* \, dV, \qquad H = \iiint_V \theta U_3^* \, dV.$$

The second integral on the right-hand side of Eq. (31) is, by Green's first theorem, equal to

$$\iint_S m_i P U_i^* \, dS,$$

which is zero because on the solid boundary U_i^* is zero and the quantity PU_i^* is periodic in x_3, the volume V being a cellular space of the convection. The third integral on the right-hand side is, for the same reasons and by virtue of Green's second theorem,

$$-\iiint (\text{grad } U_i)(\text{grad } U_i^*) \, dV = -J_1 \quad (\text{say}).$$

Thus Eq. (31) can be written as

$$\sigma J_0 + \nu J_1 = g\alpha H. \tag{32}$$

If now Eq. (11) is multiplied by T'^* and integrated, one has, by Green's second theorem and because of periodicity and the insulation of the wall,

$$\sigma I_0 + \kappa I_1 = -\beta H^* \tag{33}$$

in which

$$I_0 = \iiint_V \theta\theta^* \, dV, \qquad I_1 = \iiint_V |\text{ grad } \theta \,|^2 \, dV.$$

From Eqs. (32) and (33) it follows that

$$-\beta(\sigma^* J_0 + \nu J_1) = g\alpha(\sigma I_0 + \kappa I_1),$$

or

$$\sigma_r(g\alpha I_0 + \beta J_0) + g\alpha\kappa I_1 + \beta\nu J_1 = 0. \tag{34}$$

$$\sigma_i(g\alpha I_0 - \beta J_0) = 0. \tag{35}$$

From Eq. (35) one concludes that if σ_i is not zero β must be positive. If so, from Eq. (34) σ_r must be negative and the fluid stable—as is also to be expected from the physical point of view. Thus, under the restrictions of periodicity in the vertical direction and of infinitesimal disturbances, the principle of exchange of stabilities is valid for a viscous fluid contained in an insulated tube and heated from below.

If there are two horizontal planes intersecting the tube just considered, which are kept at constant temperatures to create a temperature gradient β, periodicity in the x_3-direction is no longer necessary for the proof, because the integral involving the pressure in Eq. (31) now vanishes on account of the vanishing of the velocity components on a solid boundary. Since the temperature fluctuations on the horizontal plates are zero, the proof presented in the last paragraph remains valid.

3. Solution of the differential system governing stability. The differential system consisting of Eqs. (20), (21), and (23) will now be solved. Since the principle of exchange of stabilities is valid, for neutral stability one needs only to consider the system (with $h = R\theta$).

$$(D^2 - a^2)h = R\,Df, \tag{36}$$

$$(D^2 - a^2)^2 f = Dh, \tag{37}$$

$$f = 0 \quad \text{and} \quad Df = 0 \quad \text{at} \quad x = \pm 1, \tag{38}$$

$$Dh = 0 \quad \text{at} \quad x = \pm 1. \tag{39}$$

Differentiating Eq. (36) and substituting Eq. (37) into the result, one has

$$(D^2 - a^2)^3 f = R\,D^2 f, \tag{40}$$

with the boundary conditions

$$f = 0, \qquad Df = 0, \qquad (D^2 - a^2)^2 f = 0 \quad \text{at} \quad x = \pm 1. \tag{41}$$

Equations (40) can be written in the form

$$(L^3 - RL - Ra^2)f = 0, \qquad (L \equiv D^2 - a^2). \tag{42}$$

The solution of the indicial equation

$$m^3 - Rm - Ra^2 = 0$$

is

$$m = \left(n + \frac{1}{n}\right)\left(\frac{R}{3}\right)^{1/2}, \tag{43}$$

in which

$$n^3 = \frac{3a^2(3/R)^{1/2} \pm (27R^{-1}a^4 - 4)^{1/2}}{2}. \tag{44}$$

Only one of the two signs need be taken, and either one can be taken. If the three roots of m are denoted by

$$m_i = \omega_i^2 - a^2, \qquad (i = 1, 2, 3) \tag{45}$$

which may be complex, the solutions of Eq. (42) are exponential functions with exponents $\pm\omega_i x$. The form of Eq. (40) permits one to resolve the problem into two parts—in the one f is even, and in the other f is odd. The solutions for even f are $\cosh \omega_i x$ and those for odd f are $\sinh \omega_i x$. The secular equation obtained from the boundary conditions and determining the relationship between a and R is, if f is even,

$$\begin{vmatrix} \cosh \omega_1 & \cosh \omega_2 & \cosh \omega_3 \\ \omega_1 \sinh \omega_1 & \omega_2 \sinh \omega_2 & \omega_3 \sinh \omega_3 \\ m_1^2 \cosh \omega_1 & m_2^2 \cosh \omega_2 & m_3^2 \cosh \omega_3 \end{vmatrix} = 0 \tag{46}$$

which, though complex as it stands, is only one real equation, on account of either the conjugacy of the two complex roots of m, or the fact that the three roots are all not complex. The secular equation for odd f is similar to Eq. (46), the only difference being that the symbols cosh and sinh are exchanged.

It can be readily shown that

$$\frac{\partial \omega_i}{\partial a} \quad \text{and} \quad \frac{\partial m_i}{\partial a} \qquad (i = 1, 2, 3)$$

contain the factor a, hence vanish as a vanishes. To find the minimum Rayleigh number, one differentiates Eq. (46) or the corresponding equation for odd f with respect to a and sets the derivative dR/da to zero. Thus, for the critical condition and as far as the first derivative, R can be considered as a constant and all differentiations with respect to a can be considered as ordinary differentiations. After differentiating Eq. (46) or the corresponding equation for odd f, one obtains three determinants having one row with its three members containing the factors

$$\frac{\partial \omega_1}{\partial a}, \frac{\partial \omega_2}{\partial a}, \frac{\partial \omega_3}{\partial a} \quad \text{or} \quad \frac{\partial m_1}{\partial a}, \frac{\partial m_2}{\partial a}, \frac{\partial m_3}{\partial a},$$

respectively. Since these vanish for a equal to zero, the minimum Rayleigh number corresponds to zero wave number in the direction of gravitation—the possibility of a maximum being ruled out by physical considerations.

The task of finding the minimum Rayleigh number is now very much lightened. One may set a equal to zero forthwith and solve the differential system directly. The equation to be solved is now

$$(D^6 - R\,D^2)f = 0, \tag{47}$$

with boundary conditions

$$f = 0, \qquad Df = 0, \qquad D^4 f = 0 \quad \text{at} \quad x = \pm 1. \tag{48}$$

For antisymmetric motion (even f), the solution can be shown by a direct calculation to be

$$-\tan R^{1/4} = \tanh R^{1/4},$$

with

$$\frac{\pi}{2} < R^{1/4} < \pi.$$

Thus

$$R_{cr}^{1/4} = 2.365, \qquad R_{cr} = 31.29 \tag{49}$$

for even f. After the writer found this number, he was informed by Dr. G. K. Batchelor that Sir Geoffrey Taylor already possessed it in 1953, though he never published it. The equation leading to this number was also found by Ostrach for different boundary conditions.

For symmetric motion (odd f), the solution is

$$\tan R^{1/4} = \tanh R^{1/4}$$

with

$$\pi < R^{1/4} < 3\pi/2.$$

Thus, approximately,

$$R_{cr}^{1/4} = \frac{5\pi}{4} = 3.927, \qquad R_{cr} = 237.6 \tag{50}$$

which, it must be remembered, is based on half of the spacing of the plates. This Rayleigh number was given specifically as the critical one for the stated problem by Ostrach

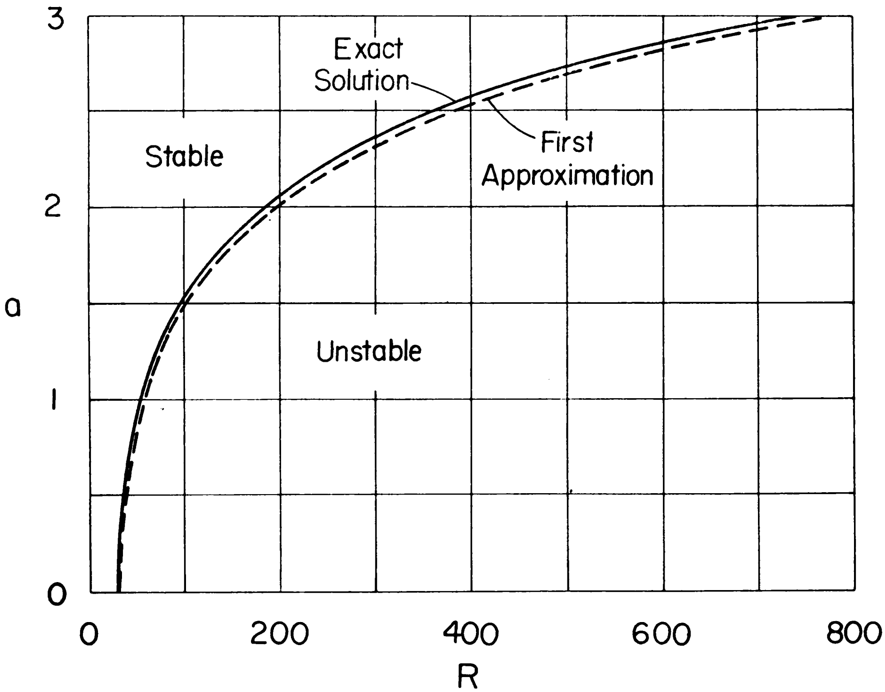

FIG. 1. Approximate and exact neutral-stability curves for antisymmetric convection between plane boundaries.

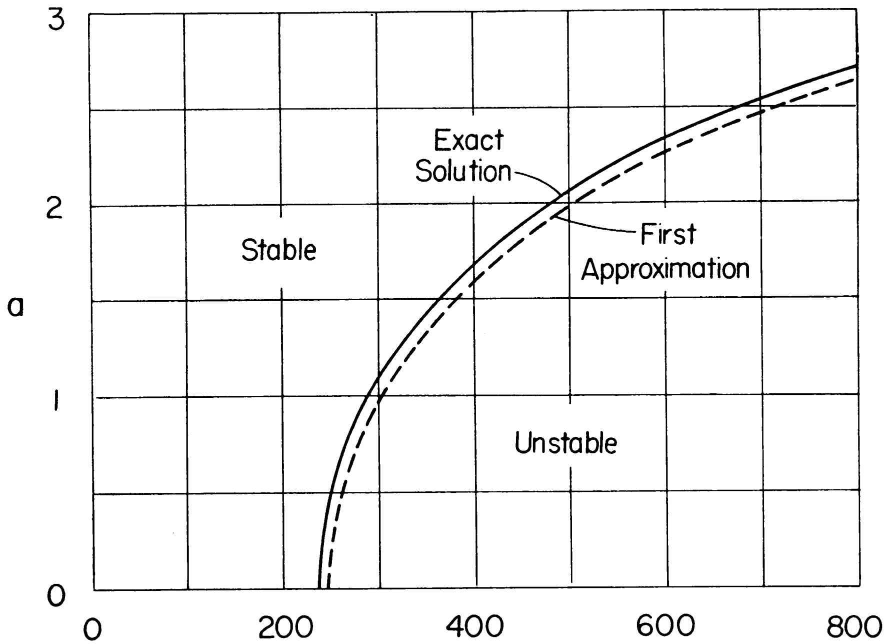

FIG. 2. Approximate and exact neutral-stability curves for symmetric convection between plane boundaries.

[13], who considered only symmetric motion. Since it is greater than the critical Rayleigh number 31.29 for even f, the stability is governed by the latter.

For non-zero values of a, the differential system consisting of Eqs. (36) to (39) will be solved first by Chandrasekhar's method [7]. The results are shown in Figs. 1 and 2 by dotted lines, for anti-symmetric and symmetric motions, respectively. After the approximate solutions are available as guides, Eqs. (45) and (46) are solved by trial and error. The results are given in Table 1. The corresponding curves are drawn in Figs. 1 and 2 for comparison with the approximate solutions. The power of Chandrasekhar's method is amply demonstrated.

TABLE 1

d	0	0.5	1	2.5	3
R (even f)	31.3*	37	53.5	360	742
R (odd f)	237.6*	251.5	288	670	

*Given before by Eqs. (49) and (50).

III. Stability of Fluid in a Tube

1. Formulation of the problem. In this part the stability of a viscous fluid in a tube of radius b and heated below is considered. The effect of rotation, which for a horizontal layer of fluid has been discussed by Chandrasekhar [10], will be investigated.

If (x_1, x_2, x_3) are cylindrical coordinates, the linearized forms of the equations of motion and the equation of heat diffusion are, under the assumptions stated at the beginning of Part II

$$\rho_0\left(\frac{\partial u_1}{\partial \tau} + \Omega \frac{\partial u_1}{\partial x_2} - 2\Omega u_2\right) = -\frac{\partial p'}{\partial x_1} + \rho_0 \nu\left(\Delta u_1 - \frac{u_1}{x_1^2} - \frac{2}{x_1^2}\frac{\partial u_2}{\partial x_2}\right), \tag{51}$$

$$\rho_0\left(\frac{\partial u_2}{\partial \tau} + \Omega \frac{\partial u_2}{\partial x_2} + 2\Omega u_1\right) = -\frac{1}{x_1}\frac{\partial p'}{\partial x_2} + \rho_0 \nu\left(\Delta u_2 - \frac{u_2}{x_1^2} + \frac{2}{x_1^2}\frac{\partial u_1}{\partial x_2}\right), \tag{52}$$

$$\rho_0\left(\frac{\partial u_3}{\partial \tau} + \Omega \frac{\partial u_3}{\partial x_2}\right) = -\frac{\partial p'}{\partial x_3} + \rho_0 \nu \Delta u_3 + \rho_0 g \alpha T', \tag{53}$$

in which the primes on p and T indicate perturbation quantities, the equation of state has been utilized, and ρ_0 has the same meaning as in Eq. (4). The linearized diffusion equation is

$$\frac{\partial T'}{\partial \tau} + \Omega \frac{\partial T'}{\partial x_2} + \beta u_3 = \kappa \Delta T' \tag{54}$$

in which β is the primary temperature gradient in the direction of the vertical. If now one sets

$$\left(\frac{x_1}{b}, x_2, \frac{x_3}{b}\right) = (r, \varphi, z)$$

$$\left(\frac{bu_1}{\kappa}, \frac{bu_2}{\kappa}, \frac{bu_3}{\kappa}\right) = (u, v, w)$$

$$T' = \beta b \Theta, \qquad \tau = \frac{b^2 t}{\kappa}, \qquad q = \frac{b^2 p'}{\rho_0 \kappa^2}$$

the linearized equations become, with B for $\Omega b^2/\kappa$

$$\frac{\partial u}{\partial t} + B\left(\frac{\partial u}{\partial \varphi} - 2v\right) = -\frac{\partial q}{\partial r} + Pr\left(\Delta u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right), \tag{55}$$

$$\frac{\partial v}{\partial t} + B\left(\frac{\partial v}{\partial \varphi} + 2u\right) = -\frac{1}{r}\frac{\partial q}{\partial \varphi} + Pr\left(\Delta v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right), \tag{56}$$

$$\frac{\partial w}{\partial t} + B\frac{\partial w}{\partial \varphi} = -\frac{\partial q}{\partial z} + Pr\, \Delta w - Pr\, R\Theta, \tag{57}$$

$$\frac{\partial \Theta}{\partial t} + B\frac{\partial \Theta}{\partial \varphi} + w = \Delta\Theta, \tag{58}$$

in which Δ is now dimensionless and R is the Rayleigh number based on b. The equation of continuity is

$$\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial \varphi} + \frac{\partial (rw)}{\partial z} = 0. \tag{59}$$

After the case of no rotation has been discussed the effect of rotation will then be investigated only for the rotationally symmetric case. If (as it turns out to be the case) rotation has no effect on the stability of the most unstable mode, the most important aspect of the problem is solved. For axisymmetric convection, Eqs. (55) to (59) become

$$\frac{\partial u}{\partial t} - 2Bv = -\frac{\partial q}{\partial r} + Pr\left(\Delta u - \frac{u}{r^2}\right), \tag{60}$$

$$\frac{\partial v}{\partial t} + 2Bu = Pr\left(\Delta v - \frac{v}{r^2}\right), \tag{61}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial q}{\partial z} + Pr\,\Delta w - Pr\,R\Theta, \tag{62}$$

$$\frac{\partial \Theta}{\partial t} + w = \Delta\Theta, \tag{63}$$

$$\frac{\partial(ru)}{\partial} + \frac{\partial(rw)}{\partial z} = 0, \tag{64}$$

in which

$$\Delta = D^2 + \frac{1}{r}D + \frac{\partial^2}{\partial z^2}, \qquad D = \frac{\partial}{\partial r}, \qquad R = -\frac{g\alpha\beta b^4}{\nu\kappa}.$$

Equation (64) permits the use of Stokes' stream function ψ in terms of which the velocity components are

$$u = \frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad w = -\frac{1}{r}D\psi. \tag{65}$$

Eliminating q between Eqs. (60) and (62), one has

$$\left[\frac{\partial}{\partial t} - Pr\left(\Delta - \frac{1}{r^2}\right)\right]\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) = 2B\frac{\partial v}{\partial z} + Pr\,R\,D\Theta$$

which, by virtue of Eqs. (65), can be written as

$$\left[\frac{\partial}{\partial t} - Pr\left(\Delta - \frac{1}{r^2}\right)\right]\left(\Delta - \frac{1}{r^2}\right)\frac{\psi}{r} = 2B\frac{\partial v}{\partial z} + Pr\,R\,D\Theta \tag{66}$$

since

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = \left(D\frac{1}{r}D + \frac{1}{r}\frac{\partial^2}{\partial z^2}\right)\psi = \left(\Delta - \frac{1}{r^2}\right)\frac{\psi}{r}.$$

Equations (61) and (63) can be written as

$$\left[\frac{\partial}{\partial t} - Pr\left(\Delta - \frac{1}{r^2}\right)\right]v = -2B\frac{\partial}{\partial z}\frac{\psi}{r}, \tag{67}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Theta = \frac{1}{r}D\psi. \tag{68}$$

If one tries a solution of the type

$$\frac{\psi}{r} = \Psi(r)\cos az\, e^{\sigma t},$$

$$v = V(r)\sin az\, e^{\sigma t},$$

$$\Theta = \theta(r)\cos az\, e^{\sigma t},$$

Eqs. (66) to (68) become

$$(\sigma - Pr\,L)L\Psi = 2aBV + Pr\,R\,D\theta, \tag{69}$$

$$(\sigma - Pr\,L)V = 2aB\Psi, \tag{70}$$

$$(\sigma - L')\theta = \frac{1}{r}\,Dr\Psi, \tag{71}$$

in which

$$\begin{aligned} L &= D^2 + \frac{1}{r}D - \frac{1}{r^2} - a^2 = D\frac{1}{r}Dr - a^2, \\ L' &= D^2 + \frac{1}{r}D - a^2 = \frac{1}{r}Dr\,D - a^2. \end{aligned} \tag{72}$$

The boundary conditions are

$$\Psi = 0, \qquad D\Psi = \text{finite}, \qquad V = 0, \qquad D\theta = 0 \quad \text{at} \quad r = 0, \tag{73}$$

$$\Psi = 0, \qquad D\Psi = 0, \qquad V = 0, \qquad D\theta = 0 \quad \text{at} \quad r = 1. \tag{74}$$

The differential system consisting of Eqs. (69) to (74) governs the stability against axisymmetric convection of the fluid column under rotation and heated from below. If there is no rotation, B can simply be set equal to zero.

2. The principle of exchange of stabilities. By a procedure similar to that used in Part II, it can be demonstrated that for axisymmetric motion a time-independent solution exists if

$$\frac{Pr^2a^8}{B^2R} \geq 4, \quad \text{or if} \quad \frac{Pr^3a^8}{B^2R} \geq 4$$

which means that for any given wave number a time-independent solution exists for sufficiently small Rayleigh number and sufficiently weak rotation. Furthermore it has been demonstrated* that, for zero wave number and undamped motion,

$$\sigma_i = -nB = -\frac{nb^2\Omega}{\kappa}, \tag{75}$$

in which $2\pi/n$ is the period of motion in the direction of φ. Equation (75) states that the convection pattern progresses with angular speed Ω. For axisymmetric motion, $n = 0$, and $\sigma_i = 0$ if the disturbances are undamped. Thus for neutral stability axisymmetric motion is time-independent for zero wave number.

3. Solution for the case of no rotation. Since for the case of no rotation a time-independent solution corresponding to neutral stability exists, the differential systems to be solved are (with $h = -R\theta$)

$$L^2\Psi = Dh, \tag{76}$$

$$L'h = \frac{R}{r}\,Dr\Psi, \tag{77}$$

*The demonstration is omitted to save space.

and

$$\Psi = 0, \qquad D\Psi = 0, \qquad Dh = 0 \quad \text{at} \quad r = 0, 1. \tag{78}$$

The differential equation and boundary conditions for V can be simply satisfied by taking V to be zero. Differentiating Eq. (77), one has

$$LDh = R(L + a^2)\Psi.$$

Thus Eq. (76) can be written as

$$L^3\Psi = R(L + a^2)\Psi, \tag{79}$$

with boundary conditions

$$\Psi = 0, \qquad D\Psi = 0, \qquad L^2\Psi = 0 \quad \text{at} \quad r = 0, 1.$$

Since the indicial equation is exactly the same as that preceding Eq. (43), the fundamental solutions of Eq. (79) are the Bessel functions

$$J_1(i\omega_1 r), \qquad J_1(i\omega_2 r), \qquad J_1(i\omega_3 r),$$

in which the ω's have the same values as in Part II (with the Rayleigh number based on b, of course). The secular equation obtained from the boundary conditions is

$$\begin{vmatrix} J_1(i\omega_1) & J_1(i\omega_2) & J_1(i\omega_3) \\ \omega_1 J_0(i\omega_1) & \omega_2 J_0(i\omega_2) & \omega_3 J_0(i\omega_3) \\ m_1^2 J_1(i\omega_1) & m_2^2 J_1(i\omega_2) & m_3^2 J_1(i\omega_3) \end{vmatrix} = 0, \tag{80}$$

since

$$\frac{dJ_1(i\omega r)}{dr} = i\omega J_0(i\omega r) - \frac{1}{r} J_1(i\omega r).$$

The determinant equation in which the ω's contain R and a, is a single equation though in the form given it is complex, and is the solution of the problem. With precisely the same arguments as in Part II, one concludes that the critical Rayleigh number occurs at zero wave number.

For zero wave number a, the secular equation can be shown by a direct calculation to be

$$J_1(R^{1/4})J_0(R^{1/4}i) + iJ_0(R^{1/4})J_1(R^{1/4}i) = 0,$$

the first root of which is

$$R^{1/4} = 4.611, \quad \text{or} \quad R = 452.1. \tag{81}$$

This number was first given by Hales [11], who considered only the axisymmetric case which, as will be seen, does not correspond to the true critical Rayleigh number.

Although only for the axisymmetric case has it been proved that the most unstable condition is associated with a wave number of zero, this situation can be expected to hold even for the other modes of motion. If one investigates the stability at zero wave number for the other modes (not axisymmetric), the conclusion reached in Sec. 2 of Part II enables one to write Eqs. (62) and (63) as

$$L_n W = R\theta, \tag{82}$$

$$L_n \theta = W, \tag{83}$$

in which

$$w = W(r) \cos n\varphi, \qquad \Theta = \theta(r) \cos n\varphi, \qquad L_n \equiv D^2 + \frac{1}{r} D - \frac{n^2}{r^2}$$

since one may assume the z-gradient of q (the perturbation pressure term) to be zero. This assumption can be justified physically from the symmetry of the flow, which is upward as well as downward and is motivated as much by the buoyancy of the hotter fluid as by the negative buoyancy of the colder fluid. The boundary conditions are

$$W = 0, \qquad D\theta = 0 \quad \text{at} \quad r = 1.$$

$$W \quad \text{and} \quad \theta \quad \text{non-singular at} \quad r = 0.$$

From Eqs. (82) and (83) it follows that

$$L_n^2 W = RW,$$

the adequate solutions of which are

$$J_n(R^{1/4}r) \quad \text{and} \quad i^n J_n(iR^{1/4}r),$$

a combination of which is to satisfy the boundary conditions

$$W = 0, \qquad DL_n W = 0 \quad \text{at} \quad r = 1.$$

The secular equation is

$$J_n(iR^{1/4})J_n'(R^{1/4}) + iJ_n'(iR^{1/4})J_n(R^{1/4}) = 0, \tag{84}$$

in which

$$J_n'(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x),$$

the primes denoting differentiation with respect to the entire argument. The equation preceeding Eq. (81) can be obtained from Eq. (84) by taking n to be zero. The first roots of Eq. (84) for integral values of n are given in Table 2. Higher roots for each n

TABLE 2

n	0	1	2
$R^{1/4}$	4.611	2.871	4.259
R	452.1	67.9	329.1

can be found, which undoubtedly correspond to neutral stability. In this case, as in the case of plane boundaries, what happens as these higher roots are crossed has not yet been rigorously investigated mathematically or understood physically. As far as the first roots go, the second mode ($n = 1$) is the most unstable, and axisymmetric disturbances are more stable not only than those of the second mode but also than those of the third mode ($n = 2$). The second mode is antisymmetric, and compared with the antisymmetric motion for the case of plane boundaries is more stable. This is not surprising, because the hydraulic radius for the latter case is, for $b = d$, exactly twice the hydraulic radius for the circular tube. The number 67.9 was first given by Taylor [12] without proof.

For non-zero wave numbers, the method of Chandrasekhar has been employed to solve the system consisting of Eqs. (76) to (78). The results are shown in Fig. 3 by the

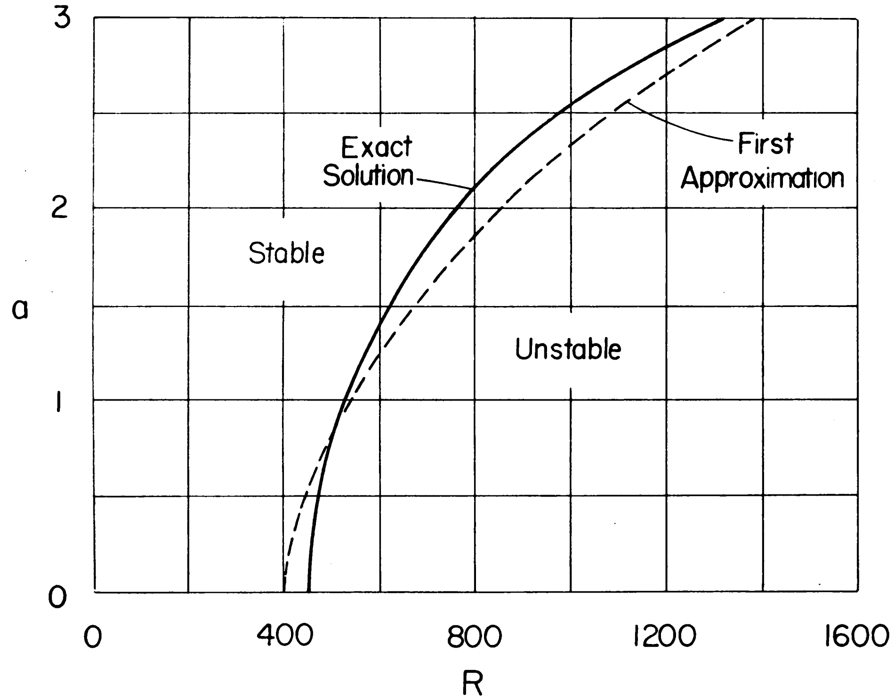

FIG. 3. Approximate and exact neutral-stability curves for axisymmetric convection in a circular tube.

dotted line. With the assistance of these results Eq. (80) is solved numerically. The results are given in Table 3 and are represented by a curve in Fig. 3. These values are in good agreement with those given by Hales [11] for a smaller range. Hales observed from his numerical data for the symmetric case but without analytic proof that the most unsteady mode corresponds to zero wave number.

TABLE 3

a	0	1	2	3
R	452.1	528	759	1322

4. Axisymmetric motion with imposed rotation. Although the time-independence at neutral stability has been demonstrated only under certain restrictions when a general rotation is present, indications for it are so strong that it can be assumed. Putting σ

equal to zero and eliminating V and θ among Eqs. (69) to (71), one has, for axisymmetric motion,

$$\left[L^3 + R(L + a^2) + \left(\frac{2aB}{Pr}\right)^2\right]\Psi = 0.$$

By a process entirely the same as that used in Part II, one can show that for any definite value of B/Pr the critical Rayleigh number occurs at zero wave number. Since the effect of rotation is manifested in the number $(aB/Pr)^2$ which contains a^2, for zero wave number the general rotation has no influence whatever on the stability. The critical Rayleigh number is therefore still given by Eq. (81). Physically, this is understandable because at zero wave number there is no radial motion, which alone is inhibited by rotation. In fact, by similar reasoning one may conjecture that a vertical magnetic field has no influence whatever on the value of the critical Rayleigh number, since no magnetic lines are crossed at zero wave number.

For non-zero wave numbers Chandrasekhar's method will be extended to be used for three equations instead of two. If $V' = (2aB/Pr)V$, $\theta' = R\theta$, the equations to be solved are

$$L^2\Psi = -V' - D\theta' \tag{85}$$

$$LV' = -\left(\frac{2aB}{Pr}\right)^2\Psi, \tag{86}$$

$$L'\theta = -R\left(\frac{1}{r}\, Dr\Psi\right), \tag{87}$$

with boundary conditions

$$V' = 0 \quad \text{at} \quad r = 0, 1; \tag{88}$$

$$\Psi = 0, \qquad D\Psi = 0 \quad \text{at} \quad r = 0, 1; \tag{89}$$

$$D\theta' = 0 \quad \text{at} \quad r = 0, 1. \tag{90}$$

If the λ's are the zeros of the Bessel function $J_1(\lambda)$, the forms

$$\theta' = \sum_{m=1}^{\infty} A_m J_0(\lambda_m r), \tag{91}$$

$$V' = \sum_{m=1}^{\infty} A'_m J_1(\lambda_m r), \tag{92}$$

are appropriate from the standpoint of the boundary conditions on V' and θ'. If these expressions are substituted in Eq. (85) and the result is solved exactly together with the boundary conditions, one obtains,

$$\begin{aligned}\Psi = \sum_{m=1}^{\infty} A_m\left[B_m iJ_1(iar) + C_m rJ_0(iar) + \frac{\lambda_m}{M^2}\, J_1(\lambda_m r)\right]&\\ + \sum_{m=1}^{\infty} A'_m\left[B'_m iJ_1(iar) + C'_m rJ_0(iar) - \frac{1}{M^2}\, J_1(\lambda_m r)\right]&\end{aligned} \tag{93}$$

in which

$$(B_m\,,B'_m) = (C_m\,,C'_m)\,\frac{iJ_0(ia)}{J_1(ia)}\,,\qquad (C_m\,,C'_m) = (\lambda_m\,,-1)\,\frac{J_0(\lambda_m)G}{M^2}\,,$$

$$M^2 = \lambda_m^2 + a^2,\qquad G^{-1} = \frac{ia^2J_0^2(ia)}{J_1(ia)} - 2J_0(ia) + iaJ_1(ia).$$

Substituting Eq. (93) in (87) and expanding in a series of $J_0(\lambda_m r)$, one has

$$-\frac{M}{R}\,A_n = \sum_{n=1}^{\infty}\left\{A_m\left[\frac{-a^2B_mP_m}{N_{0m}} + \frac{C_mQ_m}{N_{0m}} - \frac{\lambda_m^2}{M^2}\right] + A'_m\left[\frac{-a^2B'_mP_m + C'_mQ_m}{N_{0m}} + \frac{\lambda_m}{M}\right]\right\},\tag{94}$$

with

$$P_m = \frac{iJ_0(\lambda_m)J_1(ia)}{M}\,,$$

$$Q_m = -2aP_m + a\int_0^1 ir^2J_1(iar)J_0(\lambda_m r)\,dr.$$

Substituting Eq. (93) in (86) and expanding in a series of $J_1(\lambda_m r)$, one obtains

$$\begin{aligned}M\left(\frac{Pr}{2aB}\right)^2 A'_n = \sum_{m=1}^{\infty} A_m\left[\frac{-\lambda_mB_mP_m}{N_{1m}} + \frac{C_mS_m}{N_{1m}} + \frac{\lambda_m^2}{M^2}\right]&\\ + \sum_{m=1}^{\infty} A'_m\left[\frac{-\lambda_mB'_mP_m + C'_mS_m}{N_{1m}} - \frac{1}{M^2}\right]&\end{aligned}\tag{95}$$

in which

$$S_m = \int_0^1 r^2J_0(iar)J_1(\lambda_m r)\,dr,$$

$$N_{1m} = -\tfrac{1}{2}J_0(\lambda_m)J_2(\lambda_m).$$

Taking m and n to be 1 in Eqs. (94) and (95) and demanding that A_1 and A'_1 not be both zero, one arrives at the condition

$$\chi_1\Phi_2 = \chi_2\Phi_1\tag{96}$$

in which

$$\chi_1 = \frac{M_1}{R} + \frac{C_1Q_1 - a^2B_1P_1}{N_{01}} - \frac{\lambda_1^2}{M_1^2}\,,$$

$$\chi_2 = \frac{C'_1Q_1 - B'_1P_1}{N_{01}} + \frac{\lambda_1}{M_1^2}\,,$$

$$\Phi_1 = \frac{C_1S_1 - \lambda_1B_1P_1}{N_{11}} + \frac{\lambda_1^2}{M_1^2}\,,$$

$$\Phi_2 = -M_1\left(\frac{Pr}{2aB}\right)^2 + \frac{C'_1S_1 - \lambda_1B'_1P_1}{N_{11}} - \frac{1}{M_1^2}.$$

Equation (96) represents the first approximation to the relationship between a, R, and $(B/Pr)^2$ for neutral stability. Its graph for any fixed value of B/Pr intersects the neutral curve for no rotation at the R-axis, and can be expected to lie to the right of the latter curve for non-zero values of a.

IV. Conclusions

From the foregoing it can be concluded that

(1) With no rotation, convection at neutral stability is time-independent. With rotation the time-independence of axisymmetric motion for neutral stability can be proved only under certain conditions, but is valid if the wave number is zero. For other modes of motion in a rotating circular tube the undamped motion is time-independent relative to a frame of reference rotating with the tube, provided the wave number is zero.

(2) With no rotation the critical Rayleigh numbers occur at zero wave number for plane boundaries and for axisymmetric motion, and can be expected to occur at zero wave number for other modes of motion in a circular tube. Antisymmetric modes are more unstable. For such modes the critical Rayleigh number is 31.3 for plane boundaries, and 67.9 for a circular boundary. (The priority of these numbers belongs to Taylor.)

(3) Detailed relationships between the Rayleigh number and the wave number at neutral stability are given by Eq. (46), one similar to it, and Eq. (80), and by the graphs and tables. An approximate relationship connecting the Rayleigh number, rotation parameter, and wave number is given by Eq. (96) for neutral stability in the presence of rotation. Since the effect of rotation on the mean orientation of the fluid has been neglected in this paper, the results concerning the effects of rotation are only approximate. The results for zero rotation are rigorous.

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