# QUARTERLY

OF

# APPLIED MATHEMATICS



VOLUME XVII APRIL · 1959 NUMBER 1

Second-class postage paid at Providence, Rhode Island, and at Richmond, Virginia.

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

## CONTENTS

# BOOK REVIEWS

*The physics of fluids.* Edited by F. N. Frenkiel and a board of associate editors. Published by the American Institute of Physics. Vol. 1, No. 1 (Jan.-Feb. 1958). Subscription Price: $10.00 (U. S. and Canada), $11.00 (elsewhere), per annual volume of six issues.

The new periodical will be devoted to original contributions to kinetic theory, statistical mechanics, structure and general physics of gases, liquids, and other fluids, including magneto-fluid dynamics, ionized fluid and plasma physics, rarefied gases and upper atmosphere phenomena, and liquid state physics and super-fluidity. Only original papers containing significant new results will be considered appropriate for the journal, but not purely expository papers. In addition to the full-length papers of the main body of the journal, there will be shorter Research Notes, as well as Comments and Errata. Whereas the new journal starts as a bimonthly, it is expected to become a monthly in the not too distant future.

The first issue contains the following papers: "On the theory of the bubble chamber" by F. Seitz; "Wall effects in shock tube flow" by R. J. Eurich and D. B. Wheeler, Jr.; "Effect of radiation on shock wave behavior" by R. E. Marshak; "Hydromagnetic stability of a conducting fluid in a circular magnetic field" by F. N. Edmonds, Jr.; "Statistical behavior of a reacting mixture in isotropic turbulence" by S. Corrsin; "On the diffusion of a chemically reactive species in a laminar boundary layer flow" by P. L. Chambré and J. D. Young; "Quantum statistics of interacting particles; general theory and some remarks on properties of an electron gas" by E. W. Montroll and J. C. Ward.

*Esercizi Di Meccanica Razionale.* By Bruno Finzi and Paolo Udeschini. Libreria Editrice Politecnica Cesare Tamburini, Milano, 1958. viii + 544 pp.

In the third edition of this collection of problems of rational mechanics for science and engineering students at the University of Milan, the authors have gathered a more complete set of examples taken from statics, kinematics and dynamics. The problems range in difficulty from the most elementary used in an introductory lecture to the fairly complicated encountered in the various fields of solid and fluid mechanics.

All problems are entirely solved in literal form and are plane problems. No numerical examples are given.

The book is a companion volume for the treatise on Rational Mechanics of Dr. Bruno Finzi, published by Zanichelli, Bologna, in 1950.

Mario G. Salvadori

*Integral equations.* By F. G. Tricomi. Interscience Publishers, Inc., New York and London, 1957. viii + 238 pp. $7.00.

In the opening lines of his preface, the author expresses himself as follows: "One of the first subjects in mathematics to attract my attention was integral equations; yet this book appears after a score of others. Why is this? It is because the writing of a book on integral equations is a rather difficult task, a task for which many years of meditation are necessary. In fact, such a book must satisfy two requirements which are not easily reconciled. In order to facilitate theoretical applications to existence proofs it must present the main results of the theory with adequate generality and in accordance with modern standards of mathematical rigor. On the other hand, it must not be written so abstractly as to repel the physicist, engineer, and technician who certainly need and deserve this mathematical tool." It is the reviewer's firm belief that our gifted author has admirably succeeded in satisfying both requirements, and that this book is destined to take its rightful place among "the score of others" which the mathematical world has already been fortunate to receive from his pen.

There are four long chapters, two brief appendices (one on algebraic systems of linear equations and the other on Hadamard's determinant theorem) plus a useful collection of exercises, which is

(*Continued on p. 94*)

# BOOK REVIEWS

(*Continued from p. 66*)

placed at the end of the volume. Chapter I is entitled "Volterra equations". Topics treated include Volterra's integral equation of the second kind: $f(x) - \lambda \int_0^x K(x, y)f(y)\, dy = g(x)$; Volterra's integral equation of the first kind: $\int_0^x K(x, y)f(y)\, dy = g(x)$; relationship between Volterra integral equations and linear differential equations of the type

$$\frac{d^n u}{dx^n} + a_1(x)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_n(x)u = F(x);$$

integral equations of the Faltung ("closed cycle") type, where the kernel $K(x, y) = k(x - y)$; Volterra equations with kernels of the type $K(x, y) = F(x, y)/(x - y)^\alpha$, and non-linear Volterra equations. Of particular importance are the applications to the asymptotic evaluation of Bessel functions for large values of the argument and also to the more difficult asymptotic evaluation of Bessel functions whose order and whose argument are almost equal. Chapter II bears the title: "Fredholm equations." It begins by the solution of the Fredholm integral equation of the second kind; $f(x) - \lambda \int_0^1 K(x, y)f(y)\, dy = g(x)$, which leads to Neumann's series; together with the application to a particularly simple example. The Fredholm theorem for general $L_2$ kernels is proved by using E. Schmidt's idea of using the Neumann series to pass from "degenerate" (Pincherle-Goursat) kernels $K(x, y) = \sum_{k=1}^n X_k(x)Y_k(y)$ to the general kernel. Next follow Fredholm's formulas and a section on the numerical solution of integral equations. The chapter concludes with the Fredholm solution of the Dirichlet problem for a plane domain whose boundary curve has a tangent and finite curvature at each point. Chapter III deals with symmetric kernels $K(x, y) = K(y, x)$, and orthogonal systems of functions. Among other topics, one finds the Gram-Schmidt orthogonalization process; the Riesz-Fischer theorem; the bilinear formula for a kernel $K(x, y)$ in terms of its eigenfunctions and eigenvalues; the Hilbert-Schmidt expansion theorem and its applications; and Mercer's theorem on positive kernels. In the section on the convergence of the Neumann series, the reviewer missed A. Weinstein's simple but little known criterion for convergence, (see Sitzungs. Berlin Math. Gess. **26**, 1927, p. 168) namely that

$$|\lambda| \leqq \left[\max_x \int_0^1 |K(x, y)|\, dy\right]^{-1},$$

which compares favorably with C. Neumann's:

$$|\lambda| \leqq [\max_{(x,y)} |K(x, y)|]^{-1};$$

and is independent of E. Schmidt's

$$|\lambda| \leqq \left[\int_0^1 \int_0^1 K^2(x, y)\, dx\, dy\right]^{-1/2}.$$

Chapter IV is entitled: "Some types of singular or non-linear integral equations", and deals in part with theories which are still in the process of formation, as the author states. The author returns here to "Cauchy principal value integrals" and presents many of his own contributions. Of particular interest to applied mathematicians is the section devoted to the finite Hilbert transformation and the airfoil equation:

$$f(x) = \frac{1}{\pi}\int_{-1}^1 \frac{g(y)}{y - x}\, dy,$$

the integration being over $-1 < y < 1$.

In closing, an example of the author's keen concern for clarity might be mentioned. On page 130, after the definition of the jump discontinuity of the Green's function, $G$, there is a dagger, referring to a footnote at the bottom of the page, which reads as follows: "Be careful. Some authors (including myself in [48]) consider this jump with opposite sign and this implies a change of $G$ into $-G$."

J. B. Diaz

(*Continued on p. 106*)

# BOOK REVIEWS

(*Continued from p. 94*)

*An introduction to the theory of random signals and noise.* By Wilbur B. Davenport, Jr. and William L. Root. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1958. ix + 393 pp. $10.00.

Let us begin by complimenting the authors upon having written a very clear and readable introduction to the mathematical theory of random signals and noise. By writing in a leisurely and lucid fashion, and by means of a host of well chosen examples, they provide an almost painless entry into one of the most significant fields of modern analysis.

A survey of the chapters will indicate the scope of the book.

The first chapter is introductory. The next five chapters provide the background of probability theory required by the reader, elementary aspects, random variables, probability distributions, random processes and statistical averages. There is a discussion of sampling and the central limit theorem, followed by a chapter devoted to spectral analysis.

The next chapter is devoted to a discussion of shot noise in vacuum tubes and contains a detailed analysis of the physical origin of many of the problems treated in the book. This is a valuable addition to the mathematical treatment.

The following two chapters are devoted to Gaussian processes and linear systems. The problem of determining optimal linear filters leads to a discussion of the Wiener-Kolmogoroff theory of statistical detection of signals.

The only objection of any consequence that we wish to make is to the use of the "impulse function." If the same notation is desired, it is best perhaps to refer to the distribution functions of L. Schwartz. Otherwise, the Riemann-Stieltjes integral can be used.

RICHARD BELLMAN

*Theorie der Beugung Elektromagnetischer Wellen.* By W. Franz. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. 123 pp. $5.20.

This monograph, No. 4 of the series "Ergebnisse der angewandten Mathematik", gives a comprehensive treatment of the solution of Maxwell's equations for the boundary conditions appropriate to diffraction problems. It is divided into three parts. The first part discusses general mathematical approaches, introduces the Green's dyadic as analog of the Green's function of the scalar field, and formulates the needed boundary conditions. The second part treats diffraction by bodies with smooth surfaces, deriving the exact solutions for the cylinder and sphere in detail and indicating the application of the integral-equation method to diffraction at a weakly curved surface of arbitrary shape. The last part deals with diffracting bodies having sharp edges. Sommerfeld's exact solutions for the perfectly conducting wedge and halfplane are derived and approximate methods for the treatment of diffraction at slits, apertures, and circular disks are discussed. A full bibliography aids the reader in locating approaches to the diffraction problem not covered by the text. The book constitutes a valuable addition to the literature on the mathematical theory of diffraction by an author who has contributed liberally to the subject.

E. G. RAMBERG

*Mechanical resolution of linguistic problems.* By Andrew D. Booth, L. Bandwood, and J. P. Cleave. Academic Press, Inc., New York, and Butterworths Scientific Publications, London, 1958. vii + 306 pp. $9.80.

This volume, the senior author of which is one of the pioneers in the field of machine translation, is, largely, an account of experiments at the Birkbeck College Computation Laboratory in the application of digital calculators to linguistic problems. The machine used was the APEXC all-purpose electronic computer which is a laboratory model of the somewhat more versatile HEC General Purpose Electronic

Computer, a business machine produced by the British Tabulating Machine Company. The input on the APEXC is in the form of a punched paper tape, and the linguistic information is encoded in binary form, with the international teletype code utilized for this purpose. The authors recognized that the teletype code is not the most suitable one for application to linguistic analysis but adhered to it for practical reasons, mainly because ordinary teletype equipment could be attached to the machine in such a case, and the input and output procedures thus facilitated. The storage capacity of the computer, which takes the form of a magnetic drum, is 8,192 words of 32 binary digits each. The limitation of word length to 32 binary digits, i.e. to no more than six letter words represented in the five digit binary code, means, in practice, that more than one storage location must be utilized for the many longer words of a language. The APEXC can perform basically the same operations as the other standard computers.

The book contains twelve chapters and subject and name indices. It lacks, unfortunately, an annotated bibliography, although footnotes to chapter one, which is a brief history of computer application to linguistic problems, contain references to the most important works on the subject. Chapter two gives some basic information about the nature of calculating and data processing machines and the following chapters turn to the specific linguistic problems in which such machines may find suitable application. The authors point out that there are, besides translations, many other important problems of linguistic analysis for which the computer may be utilized to advantage. Some of these problems and their programming are discussed: the word counts and concordances, the analysis of sentence structure, problems of lexicography and syntactical investigation, and literary dating such as, for example, the Platonic chronology. In chapter six, the authors turn their attention to mechanical translation and its programming. The complications which may arise in such analyses, such as the treatment of stems and endings, compound words, multiple meanings, or idioms, are discussed but not always thoroughly explored. Chapter seven examines the possibility of mechanical transcription of Grade II Braille (the abbreviated form which uses special symbols for frequently occurring groupings of letters). Technically, such transcription is found feasible in spite of some ambiguities, but, because of the low output speed, it appears impractical economically. Chapters eight and nine suggest experimental schemes for the translation of French and German respectively, while the following chapter gives a brief account of the work of the Academy of Sciences of the USSR on mechanical translation of English into Russian. The question of multilingual translations is very briefly mentioned in chapter 11; the final chapter contains suggestions of technical details of a machine which would be designed specifically for translations. It emphasizes that some operations of presently available computers would be unnecessary (multiplications), while others, now lacking, should be added, such as greater word length capacity. The all-important question of adequate storage capacity is duly stressed, and a table of a proposed instruction code for such a linguistic machine is included.

The main value of this book, in the opinion of this reviewer, lies in the clarity with which it summarizes some of the basic problems concerning the application of digital calculators to linguistic problems and shows, in proper perspective, the limitations of the presently available machines. However, not all parts of the book will be equally informative for all potential readers. The linguist who may be drawn to the book will often be left puzzled by the lack of clarity of the programming descriptions, and perhaps somewhat skeptical of the way in which difficult linguistic problems have been glossed over. The analysis of French and German would have profited from the application of some pertinent principles and techniques of structural linguistics. Neither will the reader interested in details of programming on the APEXC find it in the present volume; instead, he is referred to another book which offers this information.

Henry Kucera

*Proceedings of the Third Midwestern Conference on Solid Mechanics.* University of Michigan Press, 1957. vi + 250 pp. $5.50.

The book under review contains the following 15 papers read at the Third Midwestern Conference on Solid Mechanics, held at the University of Michigan in April, 1957.

1. *Pitching Instability of Rigid Lifting Surfaces on Viscoelastic Supports in Subsonic or Supersonic Potential Flow,* by H. H. Hilton; the investigation of single-degree-of-freedom torsional instability of rigid lifting surfaces on flexible supports. 2. *On the Motion of Thermo-Viscoelastic Solid,* by M. Lessen; equations describing small motions of a viscoelastic solid are set down from a thermodynamic point of

view and the propagation of longitudinal waves is studied. 3. *Approximate Yield Conditions in Dynamic Plasticity*, by P. G. Hodge and B. Paul; a simply supported circular cylindrical shell is subjected to a blast-type radial pressure loading, and the resulting plastic displacement is computed according to square and hexagonal interaction curves. 4. *Damping of Vibrations in Elastic Rods and Sandwich Structures by Incorporation of Additional Viscoelastic Material*, by H. I. Plass; the paper is concerned with a method of reducing the vibrational energy transmitted through a structural member (circular bar with viscoelastic layer, and sandwich panel having viscoelastic core). 5. *Thermal Stresses in an Infinite Plate of Arbitrary Thickness*, by E. L. McDowell; the problem of determination of the steady-state thermal stresses and displacements in an isotropic elastic region bounded by two parallel infinite planes is studied. 6. *Elastic Stability of Conical Shells Loaded by Uniform External Pressure*, by C. E. Taylor; an approximate solution of large deflection theory for buckling pressure for complete conical shells is obtained by Rayleigh-Ritz method. 7. *Torsion with Warping Restraint of Tapered Beams*, by M. Goulard, Hsu Lo and R. I. H. Bollard; an exact and an approximate solution for a cantilever thin-walled tube of rectangular cross section in the shape of truncated cone is derived. 8. *A Theoretical and Experimental Investigation of the Stability of a Hovering Helicopter Rotor Blades*, by J. Zvara; model tests and a consistent theoretical flutter analysis of a system having three degrees of freedom are made. 9. *Reflection and Transmission of Elastic Pulses in a Bar at a Discontinuity in Cross Section*, by E. A. Ripperger and H. Norman Abramson; reflection and transmission coefficients based upon an elementary theory and steady-state conditions are compared with experimental results. 10. *The Timoshenko Beam on an Elastic Foundation*, by S. H. Crandall; a dynamic response of Timoshenko beam supported by a massless elastic foundation to a fixed pulsating load and to a moving constant load is obtained. 11. *On Rotating Blades with Flexible Mounting*, by Hsu Lo and C. E. Danforth; static and free vibration problems of such systems are investigated. 12. *The Bending Vibrations of a Twisted Rotating Beam*, by W. P. Targoff; an analytic method is developed which permits the computation of the natural modes and frequencies of rotating beams, the results of computations are compared with experimental results. 13. *Effect of Damping on Vibration Frequencies of Simple Systems*, by L. E. Malvern; one-mass force vibrating system with viscous damping across part of spring or torsional pendulum with friction bearing are studied. 14. *Bending-Torsion Flutter Sensitivity in Incompressible and Supersonic Flow*, by A. S. Richardson; it is shown that "flutter sensitivity" (the slope of the velocity vs. structural damping curve at the flutter point) is not an exact measure of the violence (or lack of violence) of the flutter mode. 15. *Flutter of Curved Plates with Edge Compression in a Supersonic Flow*, by Y. C. Fung; the importance of initial warping and pressure differential across the plate on the panel flutter characteristics is demonstrated.

The papers contained in this book, as a rule, are of significant theoretical and applied importance. It is to be regretted that the authors are not always aware of the corresponding results published in Russian. For instance, E. L. McDowell seems to be unaware of the results obtained by G. N. Maslov (An Elastic Problem of Thermoelastic Equilibrium. Izvestia Vsesouznogo Instituta Hydrotechniki, v.23 (1938)) and C. E. Taylor seems to be unaware of the results of E. I. Grigoljuk (Nonlinear Vibrations and Stability of Shallow Columns and Shells. "Izvestia" of Academy of Sciences, USSR, Department of Technical Sciences, No. 3, 33-68 (1955); Large Deflection Instability of a Closed Sandwich Conic Shell under Uniform Normal Pressure. Inzhenernyi sbornik, v.22, 111-119 (1955)).

G. S. Shapiro

*The Scientific Papers of Sir Geoffrey Ingram Taylor*. Volume I: Mechanics of Solids. Edited by G. K. Batchelor. The University Press, Cambridge, 1958. x + 593 pp. $14.50.

The present volume contains forty-one papers on a wide range of topics in mechanics of solids. They are ordered chronologically starting with the well-known paper on the use of soap films in solving torsion problems, which was written in collaboration with A. A. Griffith and published in the Proceedings of the Institution of Mechanical Engineers in 1917, and ending with a paper on strains in crystalline aggregates that was presented at the Colloquium on Deformation and Flow of Solids in Madrid in 1955. Some papers written for Government agencies are published for the first time in this volume. These include: Notes on the "Navier Effect" (1925); Lattice Distortion and Latent Heat of Cold Work in Copper (1935); Propagation of Earth Waves from an Explosion (1940); Calculation of Stress Distribution in an Autofrettaged Tube from Measurement of Stress Rings (1941); The Plastic Wave in a Wire Extended by an Impact Load (1942); The Mechanical Properties of Cordite during Impact Stressing (with R. M. Davies, 1942); and The Distortion under Pressure of an Elliptic Diaphragm which is Clamped along its Edge (1942).

*Queues, inventories and maintenance.* By Philip M. Morse. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. ix + 202 pp. $6.50.

When a system involves an ordered process moving from stage to stage it is usually spoken of as involving *flow*. In the widest sense flow problems arise in all branches of industry; in the supply of raw materials and labour, on the production lines, in the administrative handling of documents such as accounts, in the smooth provision of information to management and in the development of the business organisation itself. The flow may concern itself with physical quantities such as water behind dams or fuel to a factory; or with services such as the issue of air passages, the repair of vehicles, and the provision of unoccupied telephone lines.

One of the standard problems in this field is to ensure that capacity can handle the average flow without causing obstruction. To have too much capacity is wasteful; to have too little frustrates the process. Somewhere there lies an optimal point which obviates serious risk of bottlenecks without undue expense. The problem is to find it. The system may be subject to random disturbance or, more generally, to stochastic elements, as when customers arrive at a service counter haphazardly or variations in weather may create sudden changes in demand for a product. Thus the flow control often incorporates an element of probability theory. The dynamic plus the stochastic element constitute the basic features of the class of studies known variously as "queuing theory," "stock control," "inventory problems" and so forth. They may be considered as a fundamentally important part of operational research.

Dr. Morse has written an excellent text on the subject. Chapter 1 deals with representation in terms of probabilities of state. This leads on naturally to frequency distributions of arrival time at a service point and of service times themselves. Chapters 3 and 4 then deal with single and multiple exponential channels (the exponential case being one of the few which can be handled with comparatively simple mathematics). Chapter 5 then considers the simulation of more complicated cases. There follow four chapters on general queuing theory. The text closes with a chapter of application to inventory control and one on the maintenance of equipment. There follow a glossary, some tables and a bibliography.

Anyone who wants an introduction to queuing theory or to operational research can hardly do better than begin with this book. The basic ideas are simply and clearly expounded with a sufficient but unpretentious mathematical content. It began, so the author tells us, as an introduction to the tables with which the work concludes, but rapidly outgrew the prefatory stage and has become a monograph in its own right. Altogether it is an admirable piece of exposition in a domain where good expository accounts are rather rare. Dr. Morse promises further volumes on computational methods and detailed solutions of some of the problems touched on in this book, and if he can maintain his present standard he will have made a notable addition to the literature of operational research for which we shall all be grateful.

M. G. Kendall

*Numerical analysis.* By Kaiser S. Kunz. McGraw-Hill Book Co., Inc., New York, Toronto, and London, 1957. xv + 381 pp. $8.00.

This is an elementary treatise, written principally for engineers, although there is an apparent and laudable attention to justification of the methods presented. Nearly half the book is devoted to interpolation, differentiation and integration, including chapters on summation of series and multivariate interpolation not usually found in textbooks; much use is made of the lozenge diagram in a manner due to the author. There is only cursory mention of the important field of Gaussian quadrature.

Two chapters deal, respectively, with starting and continuing the solution of ordinary differential equations: the methods of Taylor, Picard, Euler and Runge-Kutta serve to illustrate the former, that of Milne, principally, the latter. There is attention to the accumulation of errors. Since only fifty pages suffice for these two chapters, the selection of topics had to be drastic.

A chapter is devoted to the solution of simultaneous equations and the evaluation of determinants; there is no attempt to deal with eigenvalue problems either of systems of algebraic or of differential equations. The sections on elliptic and parabolic partial differential equations manage to impart a surprising amount of information in restricted space, but the chapter on hyperbolic equations is perhaps misnamed since the method of characteristics is not mentioned and only a simple problem of the wave-equation is solved by finite differences. Fredholm and Volterra equations are discussed in the last chapter on integral equations.

Extensive references to sources are given in footnotes. The work abounds in useful worked and unworked examples, is very clearly written and should be easy reading for anyone with a knowledge of calculus and differential equations.

WALTER F. FREIBERGER

*Analysis and control of nonlinear systems.* By Y. H. Ku. The Ronald Press Co., New York, 1958. vii + 360 pp. $10.00.

This book is a book about engineering problems written by an engineer for engineers. As such, it possesses the anticipated virtues and the expected faults.

On the positive side, it contains very detailed discussions of a number of important examples, together with graphs and tables. In addition, it is very clearly written with an extensive bibliography.

The author constantly emphasizes the necessity for dealing with nonlinear equations and the improvement in performance that can be obtained by using nonlinear properties.

On the negative side, there is the objection that the book is too much like a cookbook. Only occasionally is any effort devoted to an explanation and motivation of the mathematical techniques that are employed. Some of these, in addition, are rather clumsy. Thus, for example, the proof of the superposition principle given on page 7 makes formidable a property which is an immediate consequence of linearity of the equation. The discussion of the Blasius equation on page 206 uses a very crude method of successive approximations whereas the method presented by Weyl, *Proc. Natl. Acad. Sci.*, 1941, is much simpler and more elegant, and converges rapidly and rigorously. On page 253, the same notation is used for a function and its Laplace transform.

Despite the preceding remarks, we feel that the book could be quite useful if used in conjunction with a course given by an experienced instructor.

RICHARD BELLMAN

*Studies in the mathematical theory of inventory and production.* By K. J. Arrow, S. Karlin, and H. Scarf. Stanford University Press, Stanford, California, 1958. x + 340 pp. $8.75.

Two of the most interesting classes of dynamic programming processes, viewed from the vantage points of both analysis and application, are those of inventory control and production smoothing.

The typical inventory control process may be described in the following general terms. We possess various quantities of different items for which there is a demand of stochastic nature from time to time. Since there is a penalty of some type attached to not being able to satisfy this demand, at various stages additional quantities of these items are ordered, at costs dependent upon types and quantities ordered, the times of ordering, the rate of delivery desired, and other factors as well. The problem is to determine the ordering policies which are optimal with respect to preassigned measures of efficiency.

Production smoothing processes are of quite similar structure, with the additional feature that there may or may not be costs attached to varying rates of production and rates of ordering.

Both are representative of the kinds of multi-stage decision processes that arise in economic and industrial activities.

In order to assess the role of the present volume in the continuing research in these fields, let us briefly review the very recent history of the optimal inventory problem. In the first place, there is the recognition of this decision process as a problem worthy of attention for its own sake, apart from the larger activity within which it is always imbedded. For a history of this phase, see the book by T. Whitin, *The Theory of Inventory Management*, Princeton University Press, 1953, which also contains a discussion of various preliminary mathematical attacks upon particular aspects of the general problem.

The next phase involves the analytical formulation using the techniques of the modern approach to the analysis of stochastic processes. The classic paper of Arrow, Harris and Marschak, "Optimal Inventory Policy," *Econometrica*, vol. XIX, 1951, pp. 250-272, inaugurates the use of functional equations to describe the process.

From this point the road forks. To begin with, there are essential questions pertaining to the existence and uniqueness of solution of the nonlinear functional equations obtained in this fashion. These were first treated in the paper by Dvoretzky, Kiefer and Wolfowitz, "The Inventory Problem," *Econometrica*, vol. XX, 1952, pp. 187-222, 450-466. Only after these points are settled, can one be sure of a one-to-one correspondence between the optimal policies of the original process and the solutions of the defining equations.

Secondly, there is the basic problem of studying not only optimal policies, but also sub-optimal policies. Within a class of simple feasible policies, we wish to determine the most efficient. This road is followed by Arrow, Harris and Marschak in the paper referred to above, and the inverse problem was studied by Dvoretzky, Kiefer and Wolfowitz.

Thirdly, the functional equation can be utilized to determine the structure of optimal policies as a function of the structure of the cost and penalty functions, and also of the probability distribution occurring. This study was initiated by Bellman, Glicksberg and Gross in their paper, "On the Optimal Inventory Equation," *Management Science*, vol. II, 1955, pp. 83-104; see also R. Bellman, *Dynamic Programming*, Princeton University Press, 1957, Chapter 5.

In the book under review we find a continuation of this effort, carried out quite elegantly and in great detail, and, in addition, a continuation of the description of the outcomes of various approximate policies. This last requires the use of the tools of recent renewal and queuing theory.

One of the most interesting chapters is that devoted to time-lag processes, which is to say those in which there is an appreciable delay between ordering and delivery. Uuder the assumption that backlogs occur, it is shown that the number of state variables required to describe the process may be drastically reduced. Consequently, the results obtained by Bellman, Glicksberg and Gross in the paper cited above may be extended to this case. This furnishes a ready approximation to the solution of more complicated processes.

Turning to the study of production processes over time, a pioneering work is that of Pierre Massé, "Les Reserves et la Régulation de l'Avenir dans la Vie Économique," Paris, Hermann, 1946, Volume I, Avenir Determimé, Volume II, Avenir Aléatoire. A number of interesting processes give rise via simplified mathematical models to variational problems of the following type: "Maximize a functional of the form $J(u) = \int_0^T g(u, u', t)\,dt$, subject to combinations of constraints of the type $a_1 \leq u(t) \leq a_2$, $a_3 \leq u'(t) \leq a_4$, $\int_0^T h(u, u')\,dt \leq a_5$." The presence of the constraints renders some of these problems of far greater difficulty than might be supposed upon first observation. Let us note in passing that questions of this nature arise with great frequency in the study of current engineering control processes.

The present series of studies contains an analytic treatment of some production processes and of some related problems arising in the consideration of hydroelectric systems. References are given to other work by Hohn, Modigliani and Simon, by Koopmans, by Karush and Vaszonyi, and by Bellman, Glicksberg and Gross. A good deal of unpublished work on these problems was carried out at RAND from 1948 on by Paxson, Danskin, and Fleming, stimulated by the book of Massé referred to above. Further study of variational problems with constraints may be found in Bellman, Glicksberg and Gross, "On some Variational Problems Occurring in the Theory of Dynamic Programming," *Rendiconti del Circolo Matematico di Palermo*, Serie II, Tomo III, 1954, pp. 1-35, and in Bellman, Fleming and Widder, "Variational Problems with Constraints," *Annali di Matematica*, Serie IV, Tomo XLI, 1956, pp. 301-323.

The principal authors, as well as Beckmann, Gessford and Muth, have maintained a high mathematical level with no sacrifice of clarity. All of the studies are worthwhile starting points for further research in this fundamental domain, and the introductory expository section is particularly valuable in enabling the reader to understand the origin of the following analysis.

There are, however, some ways in which the level of the book could be raised even higher. In the first place, it would have been worthwhile to have included some computational results indicating the dependence of optimal policies, minimum costs and maximum profits upon the different parameters entering into the mathematical models. Along these lines, it might also have been helpful to point out that the functional equation technique of dynamic programming furnishes computational solutions to the variational problems described in the preceding sections in a quite simple fashion without the aid of any assumptions concerning convexity, concavity and so forth; see Chapter 9 of the book referred to previously.

Secondly, it should be emphasized that one of the most important aspects of economic control processes of this type is that of predicting the future demand on the basis of the observed past demand. We thus encounter "learning" or "correlative" processes of the type that have come into prominence with the development of sequential analysis by Wald, and by Wolfowitz, Blackwell, Girshick, Karlin, and others. This fundamental problem was attacked in relation to inventory processes in the second of the papers by Dvoretzky, Kiefer and Wolfowitz referred to above.

Considering the size of the book and the amount of material it contains, it is understandable, however, that the authors felt that a line must be drawn somewhere. Perhaps, we can look forward to a second set of studies devoted exclusively to these deeper problems. In any case, the book is certainly to be recommended to anyone working in the fields of mathematical economics and operations research.

RICHARD BELLMAN

*Vector analysis.* By Louis Brand. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1957. xiii + 282 pp. $6.00.

This book covers some of the material of the author's *Vector and Tensor Analysis,* but at a more elementary level.

L. M. MILNE-THOMSON

## FOURTH U. S. CONGRESS OF APPLIED MECHANICS—1962

### Preliminary Announcement

The Fourth U. S. National Congress of Applied Mechanics will be held on the Berkeley campus of the University of California during June 18–21, 1962.

Research workers in the theoretical and applied mechanics of solids and fluids are invited to submit papers for consideration by the Editorial Committee. Further announcements concerning the preparation of papers and deadlines for submission will be made as the Congress draws nearer.

The members of the organizing committee on the Berkeley campus are: Professor W. GOLDSMITH, *Secretary;* Professor E. V. LAITONE, *Treasurer;* Professor R. M. ROSENBERG, *Chairman of the Editorial Committee;* Professor W. W. SOROKA, *General Chairman.*

Inquiries regarding the Congress should be addressed to Professor W. Goldsmith, Secretary, Division of Mechanics and Design, University of California, Berkeley 4, California.