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DIFFRACTION OF A DIPOLE FIELD BY A UNIDIRECTIONALLY CONDUCTING SEMI-INFINITE SCREEN*

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Abstract. An exact solution is obtained for the diffraction of a dipole field by a unidirectionally conducting semi-infinite plane screen. Double Laplace transforms are applied to Maxwell's equations, and the defining conditions of the unidirectionality lead to an equation between two complex functions of two complex variables. This equation is solved by an extension of the usual function-theoretical method, and we can then express the electro-magnetic field in terms of certain complex triple integrals. These are transformed into real integrals, so that it is possible to discuss the field behavior in the neighborhood of the diffracting edge. The variation of singularity along the edge of the screen is given.

1. Introduction. Diffraction by a unidirectionally conducting body is the subject of two recent investigations: Toraldo di Francia [1] has given approximate results, based on a physical discussion, for a unidirectional screen of small diameter, while Karp [2] has obtained an exact solution for the diffraction of a plane wave by a unidirectional semi-infinite screen. Provided only the far field were of interest, well-known reciprocity considerations would suffice to extend the plane wave result to the case of dipole incidence; but a discussion of the near field, and in particular of the physically interesting variation of the fields and currents along the edge of the unidirectional screen, requires the complete solution of the diffraction problem for a dipole, and this is the problem here considered.

Important applications of the theory of unidirectional screens are: (1) to the measurement of the angular momentum of electromagnetic radiation, as in Toraldo di Francia's work (see [1, 9, 10]); (2) to microwave problems involving unidirectionally conducting components; (3) to problems of propagation over anisotropic media. In any of these cases, the problem of this paper plays the role of a canonical problem, in the sense that our results as to edge behavior for a semi-infinite screen permit (by way of Keller's geometrical theory of diffraction: see [11], and references given there to earlier work by Keller and his collaborators) the deduction of asymptotic results for a large class of unidirectional screens of finite size.

Our analysis is based on a formulation of the diffraction of a dipole field by a uni-

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directional semi-infinite screen as a Wiener-Hopf problem. Jones [3] observed that diffraction problems leading to Wiener-Hopf equations are advantageously treated by taking the transform of the differential equation before applying boundary conditions: extending Jones' method to the case of dipole incidence amounts to little more than replacing single by double transforms. The known solution [4] for a perfectly conducting screen may very easily be derived in this manner.

For the case of dipole incidence on a unidirectional semi-infinite screen, it is found after taking two successive Laplace transforms of the Maxwell equation

$$(\Delta_{xyz} + k^2)\mathbf{e} = 0,$$

that we may express the double transform of the electric field as an unknown vector function of the transform variables. Obtaining relations among the components of this vector function from the remaining Maxwell equations, and then applying the boundary condition and jump conditions derived from the unidirectionality, we show in Sec. 2 that the diffraction problem is equivalent to the solution of a single transform equation for two unknown complex functions. The transform equation also involves two independent complex variables and its solution therefore requires some modification of the usual function-theoretical considerations. This solution is derived in Sec. 3, with the result that the field components are expressed as Laplace inverses.

In Sec. 4, it is found that the originals of these inverse transforms are integrals of a type introduced by Macdonald [5], plus additional terms which it is possible to transform into certain real integrals. These results permit us to give the variation of the near field as the diffracting edge is traversed.

The results are summarized in Sec. 5, in the form of a theorem, and it is verified that all conditions of the problem are met.

2. Derivation of the transform equation. We consider diffraction by a unidirectionally conducting semi-infinite screen: $x \geq 0$, $-\infty < y < \infty$, $z = 0$, where x, y, z form a right-hand rectangular coordinate system. Let $\mathbf{e}_0, \mathbf{h}_0$ be the electric and magnetic field vectors of an incident dipole field, which we take as an electric dipole with axis perpendicular to the screen, while remarking that the method which follows is also applicable to an arbitrarily oriented dipole. Locating our dipole at (x_0, y_0, z_0) with $z_0 > 0$, we may describe the incident field by the Hertz vector $(0, 0, \Pi_z)$, where $\Pi_z = e^{-ikR}/kR$ and

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.$$

The corresponding electromagnetic field components are

$$\begin{cases} \mathbf{e}_0 = \left(\frac{\partial^2 \Pi_z}{\partial z \partial x}, \frac{\partial^2 \Pi_z}{\partial z \partial y}, \frac{\partial^2 \Pi_z}{\partial z^2} + k^2 \Pi_z \right) \\ \mathbf{h}_0 = i\omega\epsilon \left(+\frac{\partial \Pi_z}{\partial y}, -\frac{\partial \Pi_z}{\partial x}, 0 \right). \end{cases} \quad (1)$$

Time dependence $e^{i\omega t}$ is understood.

Now denote by $\mathbf{e}_0 + \mathbf{e}, \mathbf{h}_0 + \mathbf{h}$ the total electric and magnetic fields resulting from the incidence of $\mathbf{e}_0, \mathbf{h}_0$ upon the given screen. Then the scattered field vectors \mathbf{e}, \mathbf{h} satisfy the time-harmonic Maxwell equations

$$(\Delta_{xyz} + k^2)\mathbf{e} = 0, \quad (2)$$

$$\nabla \cdot \mathbf{e} = 0, \quad (3)$$

$$-i\omega\mu\mathbf{h} = \nabla \times \mathbf{e}, \quad (4)$$

subject to a set of conditions (boundary condition, two jump conditions and an edge condition) of which the first three are intended as a phenomenological description of the unidirectional conductivity of the diffracting screen. These conditions are conveniently stated in terms of field components in the direction ξ of conductivity and the direction η normal to ξ :

$$e_\xi = -e_{0\xi}, \quad (5)$$

$$[h_\xi] = h_\xi(z+0) - h_\xi(z-0) = 0, \text{ across the screen,} \quad (6)$$

$$[e_\eta] = 0, \text{ across the screen,} \quad (7)$$

$$[h_\eta] = 0, \text{ at the edge of the screen.} \quad (8)$$

We also assume that \mathbf{e} , \mathbf{h} are integrable at the edge of the screen: this, with (8), yields a unique solution.

Conditions at infinity complete our specification of the scattered field. We impose the familiar condition that \mathbf{e} , \mathbf{h} be exponentially damped solutions of the three-dimensional wave equation. The implied behavior at infinity is that of the elementary solution e^{-ikR}/kR , where we suppose k to have a negative imaginary part: $k = k_1 - ik_2$ ($k_2 > 0$).

Now define the double Laplace transforms \mathbf{E} , \mathbf{H} of the scattered field vectors \mathbf{e} , \mathbf{h} :

$$\begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{Bmatrix} \mathbf{e} \\ \mathbf{h} \end{Bmatrix} e^{-sz+ty} dx dy.$$

It follows from (2) that \mathbf{E} satisfies

$$\left(\frac{\partial^2}{\partial z^2} + K^2 \right) \mathbf{E} = 0, \quad (9)$$

where $K^2 = k^2 + s^2 + t^2$. Our condition on \mathbf{e} at infinity yields the range of validity of (9): we fix K by the choice

$$(k^2 + s^2 + t^2)^{1/2}|_{s=t=0} = +k,$$

and note that (9) is meaningful for $|\operatorname{Re} t| < k_2$, $|\operatorname{Re} s| < |\operatorname{Im}(k^2 + t^2)^{1/2}|$. In this domain of s , t we write the solution of (9) (notice $\operatorname{Im} K < 0$) as

$$\begin{aligned} \mathbf{E} &= \mathbf{A}(s, t) \exp[-iK(z+z_0)], & z > 0 \\ &= \mathbf{B}(s, t) \exp[iK(z-z_0)], & z < 0. \end{aligned} \quad (10)$$

We proceed to determine the transform of the scattered electric field by solving for $\mathbf{A}(s, t)$, $\mathbf{B}(s, t)$.

It is convenient to rotate coordinates from x , y to ξ , η . Let α_0 ($0 < \alpha_0 < \pi/2$) be the angle, measured clockwise, from the x -direction to the ξ -direction. Then

$$\begin{aligned} \xi &= x \cos \alpha_0 - y \sin \alpha_0, \\ \eta &= x \sin \alpha_0 + y \cos \alpha_0. \end{aligned} \quad (11)$$

The corresponding rotated transform variables p , q , with the property $p\xi + q\eta = sx - ty$, are given by

$$\begin{aligned} p &= s \cos \alpha_0 + t \sin \alpha_0, \\ q &= s \sin \alpha_0 - t \cos \alpha_0. \end{aligned} \quad (12)$$

We notice that $s^2 + t^2 = p^2 + q^2$ is invariant under the rotation, so that $K^2 = k^2 + s^2 + t^2 = k^2 + p^2 + q^2$. It is clear that (10) is unchanged if \mathbf{A} , \mathbf{B} are understood to be functions of p , q rather than s , t .

We now deduce the basic transform equation of our problem. Observe first that (10) permits us to write the transform of (4) as

$$-i\omega\mu\mathbf{H} = (pi_1 + qi_2 \mp iKi_3) \cdot \mathbf{E}, \quad (13)$$

where i_1 , i_2 , i_3 , are unit vectors in the ξ , η and z directions. Similarly, we write the transform of (3) as

$$(pi_1 + qi_2 \mp iKi_3) \cdot \mathbf{E} = 0, \quad (14)$$

which according to (10) is equivalent to

$$\begin{aligned} pA_\xi + qA_\eta - iKA_z &= 0, \\ pB_\xi + qB_\eta + iKB_z &= 0. \end{aligned} \quad (15)$$

Since conditions (5), (7) yield $A_\xi = B_\xi$, $A_\eta = B_\eta$, we see that $B_z = -A_z$, while

$$A_z = \frac{1}{iK} (pA_\xi + qA_\eta). \quad (16)$$

The transform of (6), with $[H_\xi]$ computed from (13), then gives

$$qA_z = -iKA_\eta, \quad (17)$$

from which

$$A_\eta = \frac{pqA_\xi}{k^2 + p^2}. \quad (18)$$

Now calculate $[H_\eta]$ from (13). If we use (17), (18) to eliminate A_z , and if we denote $(i\omega\mu/2) [H_\eta] = \Lambda$, the result is our basic transform equation:

$$(k^2 + p^2)\Lambda = k^2 iKA_\xi \exp(-iKz_0). \quad (19)$$

Now simplify (19) by introducing the boundary condition (5). Let

$$\mathbf{E}_+ = \int_0^\infty dx \int_{-\infty}^\infty dy e^{-sz+ty} \mathbf{e}(x, y, z),$$

$$\mathbf{E}_- = \int_{-\infty}^0 dx \int_{-\infty}^\infty dy e^{-sz+ty} \mathbf{e}(x, y, z),$$

$$\mathbf{E}_\pm(s, t, z)|_{z=0} = \mathbf{E}_\pm(0)$$

and define the ξ -component of $\mathbf{E}_\pm(0)$ in accordance with (11) as

$$E_{\pm\xi}(0) = E_{\pm x}(0) \cos \alpha_0 - E_{\pm y}(0) \sin \alpha_0.$$

Then

$$E_{+\xi}(0) + E_{-\xi}(0) = A_\xi \exp(-iKz_0), \quad (20)$$

where we see from (5) that

$$E_{+\xi}(0) = - \int_0^\infty dx \int_{-\infty}^\infty dy e^{-sx+ty} e_{0\xi}$$

and from (1) that

$$e_{0\xi} = \left(\frac{\partial^2 \Pi_z}{\partial z \partial x} \cos \alpha_0 - \frac{\partial^2 \Pi_z}{\partial z \partial y} \sin \alpha_0 \right) \Big|_{z=0}. \quad (21)$$

Since Π_z differs only by a factor of $k/4\pi$ from the free-space Green's function of the three-dimensional wave equation, we readily deduce the y -transform (valid for $|\operatorname{Re} t| < k_2$)

$$\begin{aligned} \int_{-\infty}^\infty \frac{\exp[-ik\{(x-x_0)^2 + (y-y_0)^2 + z_0^2\}^{1/2} + ty]}{k\{(x-x_0)^2 + (y-y_0)^2 + z_0^2\}^{1/2}} dy \\ = -\frac{\pi i}{k} e^{ty_0} H_0^{(2)}[K_0\{(x-x_0)^2 + z_0^2\}^{1/2}], \end{aligned} \quad (22)$$

where

$$K_0 = (k^2 + t^2)^{1/2},$$

and

$$(k^2 + t^2)^{1/2}|_{t=0} = +k.$$

The x -transform

$$\int_0^\infty dx e^{-sx}$$

of the right side of (22) is then obtained from the known integral representation

$$H_0^{(2)}[K_0\{(x-x_0)^2 + z_0^2\}^{1/2}] = -\frac{1}{\pi i} \int_{w_0-i\infty}^{w_0+i\infty} \frac{\exp[w(x-x_0) - i(K_0^2 + w^2)^{1/2}z_0]}{(K_0^2 + w^2)^{1/2}} dw, \quad (23)$$

where $|w_0| < |\operatorname{Im} K_0|$. Recalling that $z_0 > 0$, we conclude that the double transform of (21), valid for $\operatorname{Re} s > w_0$, is

$$E_{+\xi}(0) = \frac{ipe^{ty_0}}{k} \int_{w_0-i\infty}^{w_0+i\infty} \frac{\exp[-wx_0 - i(K_0^2 + w^2)^{1/2}z_0]}{s-w} dw. \quad (24)$$

Applying (20) and (24), we put (19) in the form

$$(k^2 + p^2)\Lambda = -kpKe^{ty_0} \int_{w_0-i\infty}^{w_0+i\infty} \frac{\exp[-wx_0 - i(K_0^2 + w^2)^{1/2}z_0]}{s-w} dw + k^2 iKE_{-\xi}(0), \quad (25)$$

where $|w_0| < |\operatorname{Im} K_0|$. Either of the two unknown functions Λ , $E_{-\xi}(0)$ completely determines E .

3. Transforms of the field components. The usual Wiener-Hopf techniques are not immediately applicable to (25), since the domains of regularity of the functions depend on the two complex variables s and t simultaneously. In this section, we show how the difficulty may be overcome by restricting all operations to a suitable range of t . For such t , (25) is treated as if s alone were the variable. Certain representation theorems

for the Laplace transform are applied to carry out the function-theoretical argument, and the resulting knowledge of Λ yields the transforms of the various field components.

Now all field components are required to be exponentially damped solutions of the three-dimensional wave equation. It follows that $\Lambda = (i\omega\mu/2) [H_+]$ is regular for $\text{Re } (s + iK_0) > 0$, while $E_{-\xi}(0)$ is regular for $\text{Re } (iK_0 - s) > 0$. The respective half-planes of regularity depend on $K_0 = (k^2 + t^2)^{1/2}$, but a discussion based on (22) shows that the same t -condition suffices for the regularity of both Λ and $E_{-\xi}(0)$, namely that t be in the strip $|\text{Re } t| < k_2$.

The factorization $K = (s + iK_0)^{1/2} (s - iK_0)^{1/2}$ then permits us to rewrite (25) as

$$kpe^{i\omega_0}(p - ik)^{-1}(s - iK_0)^{1/2} \int_{w_0 - i\infty}^{w_0 + i\infty} \exp[-wx_0 - i(K_0^2 + w^2)^{1/2}z_0](s - w)^{-1} dw \quad (26)$$

$$= -(p + ik)(s + iK_0)^{-1/2}\Lambda + ik^2(p - ik)^{-1}(s - iK_0)^{1/2}E_{-\xi}(0),$$

where $|w_0| < |\text{Im } K_0|$, and where the condition $|\text{Re } t| < k_2$ is met by choosing $\text{Re } t = 0$. Then $|\text{Im } K_0| \geq k_2$, and we may take $|w_0| < k_2$. Denote the left side of (26) by $f(s)$ and the first and second terms on the right by $g(s)$, $h(s)$ respectively. The equation

$$f(s) = g(s) + h(s), \quad (26')$$

where $f(s)$ is regular in the strip $w_0 < \text{Re } s < k_2$, and $g(s)$, $h(s)$ are regular in the overlapping half-plane $\text{Re } s > -k_2$, $\text{Re } s < k_2$, may be solved for the unknown functions $g(s)$, $h(s)$ by an application of the Wiener-Hopf technique. The procedure appears to be equivalent to that given formally by Harrington [6], but the justification may perhaps be of interest.

We reason as follows. The fact that $f(s)$ is regular in a strip suggests that it is there represented by a two-sided Laplace transform. We shall assume such a representation: the assumption will be justified on the basis of results we obtain for $g(s)$ and $h(s)$. Provisionally, then, we write

$$f(s) = \int_{-\infty}^{\infty} e^{-sz} F(x) dx, \quad (27)$$

where

$$F(x) = \frac{1}{2\pi i} \int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} e^{x\zeta} f(\zeta) d\zeta \quad (|\zeta_0| < k_2).$$

Next observe that $h(s)$ has the form

$$h(s) = ik^2 \sec \alpha_0 s^{-1/2} E_{-\xi}(0) + s^{-3/2} h_1(s),$$

where $h_1(s)$ is bounded for $\text{Re } s < k_2$. The function

$$s^{-3/2} h_1(s) = h(s) - ik^2 \sec \alpha_0 s^{-1/2} E_{-\xi}(0) \quad (28)$$

then meets all conditions of a standard representation theorem [7] for the one-sided Laplace transform. Since $ik^2 \sec \alpha_0 s^{-1/2} E_{-\xi}(0)$ is a Laplace transform [namely the transform of the convolution of the inverses of $s^{-1/2}$ and of $ik^2 \sec \alpha_0 E_{-\xi}(0)$], it follows that $h(s)$ is.

As to $g(s)$, we notice that Λ and $s^{-1/2} \Lambda$ are transforms. To consider $s \Lambda$, write

$$s\Lambda = \frac{i\omega\mu}{2} \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \frac{\partial [h_{\eta}]}{\partial x} e^{-sz} dx. \quad (29)$$

An integration by parts, with an application of our edge condition (8) as $\epsilon \rightarrow 0$, permits us to conclude that sA is a Laplace transform; an argument similar to that given for $h(s)$ then shows that $g(s)$ meets the conditions of the representation theorem [7] used above.

Our result (from which the assumed representation (27) follows directly) is that $g(s)$, $h(s)$ are Laplace transforms of certain functions $G(x)$, $H(x)$ over the ranges $(0, \infty)$, $(-\infty, 0)$ respectively. Adding these representations to the now proved (27), we replace (26') by

$$\int_{-\infty}^{\infty} e^{-sz} F(x) dx = \int_0^{\infty} e^{-sz} G(x) dx + \int_{-\infty}^0 e^{-sz} H(x) dx. \quad (30)$$

The standard uniqueness theorem [8] for two-sided transforms readily yields

$$\int_0^{\infty} e^{-sz} F(x) dx = g(s), \quad (31a)$$

$$\int_{-\infty}^0 e^{-sz} F(x) dx = h(s). \quad (31b)$$

Our transform equation (26') is therefore solved for the two unknown functions $g(s)$, $h(s)$. We remark that the familiar function-theoretic ingredients (factorization via Cauchy's integral formula, the appeal to analytic continuation and to Liouville's theorem) of the Wiener-Hopf techniques are implicit in the representation and uniqueness theorems we have used. At the same time, it appears that the implications and domain of applicability of the technique may be considerably enlarged by drawing on more general representation theorems, which are independent of the classical function-theoretic approach of Wiener and Hopf. We develop this point of view, and give applications to diffraction theory, in a forthcoming investigation.

Expressions for the transforms of the various field components follow at once from either (31a) or (31b). Let us apply (31a): the substitution of our definitions [see (26), (26')] of $g(s)$, $f(s)$ and

$$F(x) = \frac{1}{2\pi i} \int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} f(\zeta) e^{\zeta x} d\zeta \quad (|\zeta_0| < k_2)$$

in (31a) leads to

$$\Lambda = k e^{i\nu_0} (s + iK_0)^{1/2} (p + ik)^{-1} \cdot J, \quad (32)$$

where

$$J = \int_{w_0 - i\infty}^{w_0 + i\infty} \frac{(w \cos \alpha_0 + t \sin \alpha_0)(w - iK_0)^{1/2}}{(w - s)(w \cos \alpha_0 + t \sin \alpha_0 - ik)} \exp[-wx_0 - iz_0(K_0^2 + w^2)^{1/2}] dw \quad (33)$$

and $|w_0| < |\operatorname{Im} K_0|$. We then apply (19), (18), (17) in succession to find

$$A_{\xi} \exp(-ikz_0) = \frac{\exp(ty_0)(p - ik)}{ik(s - iK_0)^{1/2}} \cdot J, \quad (34)$$

$$A_{\eta} \exp(-ikz_0) = \frac{pq \exp(ty_0)}{ik(p + ik)(s - iK_0)^{1/2}} \cdot J, \quad (35)$$

$$A_z \exp(-ikz_0) = \frac{-p \exp(ty_0)(s + iK_0)^{1/2}}{k(p + ik)} \cdot J. \quad (36)$$

The x - and y - components of $\mathbf{A}(x, t)$ follow from (34), (35):

$$A_x \exp(-iKz_0) = \frac{\exp(ty_0)\{(k^2 + p^2) \cos \alpha_0 + pq \sin \alpha_0\}}{ik(p + ik)(s - iK_0)^{1/2}} J, \quad (37)$$

$$A_y \exp(-iKz_0) = \frac{-\exp(ty_0)\{(k^2 + p^2) \sin \alpha_0 - pq \cos \alpha_0\}}{ik(p + ik)(s - iK_0)^{1/2}} J. \quad (38)$$

In accordance with (10), the scattered electric field is given by

$$\mathbf{e} = \frac{1}{(2\pi i)^2} \int_S \int_T \{\mathbf{A}(s, t) \exp(-iKz_0)\} \exp[sx - i|z|K - ty] dt ds, \quad (39)$$

where S denotes a vertical contour from $s_0 - i\infty$ to $s_0 + i\infty$, with $|s_0| < k_z$, while T denotes a vertical contour from $-i\infty$ to $+i\infty$. We now substitute (34), (35), (36) in (39) to obtain explicit expressions for the field components e_ξ , e_η , e_z . Writing W for the vertical contour from $w_0 - i\infty$ to $w_0 + i\infty$, where $|w_0| < |\operatorname{Im} K_0|$, and denoting $(w \cos \alpha_0 + t \sin \alpha_0)$ by $-p_0$, we introduce the complex integrals

$$I = -\frac{1}{(2\pi i)^2 k} \int_S \int_T \int_W \frac{pp_0}{(p + ik)(p_0 + ik)} \cdot \frac{\exp[sx - wx_0 - iK|z| - i(K_0^2 + w^2)^{1/2}z_0 - t(y - y_0)]}{(s - w)\{(s - iK_0)(w + iK_0)\}^{1/2}} dw dt ds \quad (40)$$

and

$$G = \frac{-k}{(2\pi i)^2} \int_S \int_T \int_W \frac{\exp[sx - wx_0 - iK|z| - i(K_0^2 + w^2)^{1/2}z_0 - t(y - y_0)]}{(s - w)(p + ik)(p_0 + ik)\{(s - iK_0)(w + iK_0)\}^{1/2}} dw dt ds \quad (41)$$

in terms of which we express the electric field components as

$$e_\xi = I_{\xi z_0} + G_{\xi z_0}, \quad (42)$$

$$e_\eta = \frac{1}{k^2} G_{\xi \xi_0 \eta z_0}, \quad (43)$$

$$e_z = \frac{1}{k^2} G_{\xi \xi_0 z z_0}, \quad (44)$$

where the subscripts of I , G denote partial derivatives, with the ξ and ξ_0 derivatives defined by

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} \cos \alpha_0 - \frac{\partial}{\partial y} \sin \alpha_0, \quad (45)$$

$$\frac{\partial}{\partial \xi_0} = \frac{\partial}{\partial x_0} \cos \alpha_0 - \frac{\partial}{\partial y_0} \sin \alpha_0. \quad (46)$$

The operational equivalents of $\partial/\partial \xi$, $\partial/\partial \xi_0$ are accordingly p , p_0 respectively.

Applying (4), the components of the scattered magnetic field are

$$h_\xi = 0, \quad (47)$$

$$h_\eta = -\frac{1}{i\omega\mu} G_{\xi_0 z z_0}, \quad (48)$$

$$h_z = \frac{1}{i\omega\mu} G_{\xi_0 \eta z_0}. \quad (49)$$

The x - and y -components of the field are linear combinations of the ξ - and η -components:

$$\begin{aligned}e_x &= e_\xi \cos \alpha_0 + e_\eta \sin \alpha_0 \\e_y &= -e_\xi \sin \alpha_0 + e_\eta \cos \alpha_0\end{aligned}$$

and

$$\begin{aligned}h_x &= h_\xi \cos \alpha_0 + h_\eta \sin \alpha_0 \\h_y &= -h_\xi \sin \alpha_0 + h_\eta \cos \alpha_0.\end{aligned}$$

4. Discussion of the field; edge behavior. In this section, we relate the integrals I , G of (40), (41) to an integral

$$F = \frac{-1}{(2\pi i)^2 k} \int_S \int_T \int_W \frac{\exp [sx - wx_0 - iK |z| - i(K_0^2 + w^2)^{1/2} z_0 - t(y - y_0)]}{(s - w) \{(s - iK_0)(w + iK_0)\}^{1/2}} dw dt ds \quad (50)$$

which may be calculated explicitly. It is in fact readily shown that $\Phi = e^{-ikR}/kR + F$ is exactly the classical solution of Macdonald [5] for the field of a point source in the presence of a semi-infinite screen on which the field vanishes. Macdonald's result

$$\Phi = I_R - I_S, \quad (51)$$

where

$$I_R = \frac{i}{2} \int_{-\mu_R}^{\infty} H_1^{(2)}(kR \cosh \mu) d\mu, \quad (52)$$

$$I_S = \frac{i}{2} \int_{-\mu_S}^{\infty} H_1^{(2)}(kS \cosh \mu) d\mu, \quad (53)$$

with

$$\begin{aligned}R &= [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}, \\S &= [(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{1/2}, \\\mu_R &= \sinh^{-1} \{ (2/R)(rr_0)^{1/2} \cos [(\phi - \phi_0)/2] \}, \\\mu_S &= \sinh^{-1} \{ (2/S)(rr_0)^{1/2} \cos [(\phi + \phi_0)/2] \},\end{aligned}$$

and

$$x = r \cos \phi, \quad x_0 = r_0 \cos \phi_0, \quad z = r \sin \phi, \quad z_0 = r_0 \sin \phi_0,$$

will therefore facilitate our discussion of the present solution.

Now consider the definitions (40), (41), (50) of I , G , F . We find

$$\frac{\partial^2 G}{\partial \xi \partial \xi_0} = k^2 I \quad (54)$$

and

$$\left(\frac{\partial}{\partial \xi} + ik \right) \left(\frac{\partial}{\partial \xi_0} + ik \right) G = k^2 F, \quad (55)$$

so that

$$I = F + G + \frac{1}{ik} \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi_0} \right) G. \quad (56)$$

Applying (54) to \mathbf{e} as given by (42), (43), (44) we have

$$\mathbf{e} = (I_{\xi z_0} + G_{\xi_0 z_0}, I_{\eta z_0}, I_{zz_0}). \quad (57)$$

The result $F = -(e^{-ikR}/kR) + \Phi$, where Φ is Macdonald's fundamental solution as given by (51), (52), (53), is then combined with (56) to give

$$\mathbf{e} + \mathbf{e}_0 = \left\{ \Phi_{\xi z_0} + \Psi^{(1)}, \Phi_{\eta z_0} + \Psi^{(2)}, \Phi_{zz_0} + k^2 \left(\frac{e^{-ikR}}{kR} \right) + \Psi^{(3)} \right\}, \quad (58)$$

where

$$\Psi^{(1)} = G_{z_0 \xi} + G_{z_0 \xi_0} + \frac{1}{ik} (G_{z_0 \xi \xi_0} + G_{z_0 \xi \xi}), \quad (59i)$$

$$\Psi^{(2)} = G_{\eta z_0} + \frac{1}{ik} (G_{\eta z_0 \xi_0} + G_{\eta z_0 \xi}), \quad (59ii)$$

$$\Psi^{(3)} = G_{zz_0} + \frac{1}{ik} (G_{zz_0 \xi_0} + G_{zz_0 \xi}). \quad (59iii)$$

An explicit integral representation for $\Psi^{(1)}$ now follows from (41), (59i). We see first that

$$\left(\frac{\partial}{\partial \xi_0} + ik \right) \Psi^{(1)} = -i \cos \alpha_0 \frac{\partial M}{\partial z_0}$$

where

$$M = \frac{1}{(2\pi i)^2} \int_S \int_T \int_W \exp [sx - wx_0 - iK |z| - i(K_0^2 + w^2)^{1/2} z_0 - t(y - y_0)] \frac{dw dt ds}{\{(s - iK_0)(w + iK_0)\}^{1/2}}.$$

But we may integrate over W , T , S successively to evaluate M . Consider

$$M_1 = \frac{1}{2\pi i} \int_W (w + iK_0)^{-1/2} \exp [-wx_0 - iz_0(K_0^2 + w^2)^{1/2}] dw.$$

Let $x_0 = r_0 \cos \phi_0$, $z_0 = r_0 \sin \phi_0$ and apply Cauchy's theorem to deform the path into $w = iK_0 \cos(\phi_0 + i\eta)$, with $-\infty < \eta < \infty$; the result is

$$\begin{aligned} M_1 &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp(-iK_0 r_0 \cosh \eta) K_0 \sin(\phi_0 + i\eta) d\eta}{(2iK_0)^{1/2} \cos[(\phi_0 + i\eta)/2]} \\ &= \frac{e^{-i\pi/4} (2K_0)^{1/2}}{2\pi i} \int_{-\infty}^{\infty} \exp(-iK_0 r_0 \cosh \eta) \sin[(\phi_0 + i\eta)/2] d\eta \\ &= -i \sin(\phi_0/2) r_0^{-1/2} \exp(-iK_0 r_0). \end{aligned}$$

Similarly, we deform S into $s = -ik \cos(\theta + i\eta)$ to obtain

$$M_2 = \frac{1}{2\pi i} \int_S (s - iK_0)^{-1/2} \exp[sx - iK |z|] ds = i \sin(\phi/2) r^{-1/2} \exp(-iK_0 r),$$

where $x = r \cos \phi$, $z = r \sin \phi$. We then have

$$M = \int_T (M_1)(M_2) \exp [-t(y - y_0)] dt$$

$$= (rr_0)^{-1/2} \sin (\phi/2) \sin (\phi_0/2) \int_T \exp [-t(y - y_0) - iK_0(r + r_0)] dt,$$

and, applying (23),

$$M = -\pi k(r + r_0)(rr_0)^{-1/2} \sin (\phi/2) \sin (\phi_0/2) \rho^{-1} H_1^{(2)}(k\rho),$$

with $\rho = [(y - y_0)^2 + (r + r_0)^2]^{1/2}$. Solving the differential equation

$$\left(\frac{\partial}{\partial \xi_0} + ik \right) \Psi^{(1)} = -i \cos \alpha_0 \frac{\partial M}{\partial z_0},$$

we find

$$\Psi^{(1)} = r^{-1/2} \sin (\phi/2) \left\{ \pi i k \cos \alpha_0 \sin (\phi_0/2) \exp (-ik\xi_0) \right.$$

$$\left. \cdot \frac{\partial}{\partial z_0} \int_{-\infty}^{\xi_0} \frac{(r + r'_0)}{(r'_0)^{1/2}} \frac{H_1^{(2)}(k\rho')}{\rho'} \exp (ik\xi'_0) d\xi'_0 \right\}, \quad (60i)$$

where

$$x = r \cos \phi, \quad z = r \sin \phi$$

$$r'_0 = [(x'_0)^2 + z_0'^2]^{1/2}, \quad x'_0 = \xi'_0 \cos \alpha_0 + \eta_0 \sin \alpha_0$$

$$\rho' = [(y - y'_0)^2 + (r + r'_0)^2]^{1/2}, \quad y'_0 = -\xi'_0 \sin \alpha_0 + \eta_0 \cos \alpha_0.$$

It is to be noticed that $\Psi^{(1)}$ vanishes on the screen, as does $\Phi_{\xi_{**}}$, so that our boundary condition (5) is satisfied.

Operational considerations now permit us to evaluate $\Psi^{(2)}$, $\Psi^{(3)}$. Let G_* denote the image of G under our two-dimensional transform; the operational equivalences

$$\Psi^{(1)} \doteq (-1/k)(K_0^2 + w^2)^{1/2}(p + p_0 + ik)G_*,$$

$$\Psi^{(2)} \doteq (-1/k)(K_0^2 + w^2)^{1/2}q(p + p_0 + ik)G_*,$$

$$\Psi^{(3)} \doteq (i/k)\{(K_0^2 + w^2)(K_0^2 + s^2)\}^{1/2}(p + p_0 + ik)G_*,$$

follow from (59i), (59ii), (59iii). Then let $\Psi_*^{(1)}$ denote the image of $\Psi^{(1)}$; the equivalences

$$\Psi^{(2)} \doteq \frac{q}{p + ik} \Psi_*^{(1)}$$

$$\Psi^{(3)} \doteq \frac{-i(K_0^2 + s^2)^{1/2}}{p + ik} \Psi_*^{(1)}$$

lead to evaluations

$$\Psi^{(2)} = e^{-ik\xi} \int_{-\infty}^{\xi} \exp (ik\xi') \frac{\partial \Psi^{(1)}}{\partial \eta} \bigg|_{\xi=\xi'} d\xi' \quad (60ii)$$

$$\Psi^{(3)} = e^{-ik\xi} \int_{-\infty}^{\xi} \exp (ik\xi') \frac{\partial \Psi^{(1)}}{\partial z} \bigg|_{\xi=\xi'} d\xi', \quad (60iii)$$

where $\Psi^{(1)}$ is given by (60i). It is clear from (60ii) that $\Psi^{(2)}$ is an even function of z and therefore continuous across the screen: since $\Phi_{\eta\eta}$ vanishes on the screen, it follows that the jump condition (7) is satisfied.

The behavior of $\Psi^{(1)}$ in the neighborhood of the edge of the screen follows from (60i). As $r \rightarrow 0$, the behavior is

$$\Psi^{(1)} \sim r^{-1/2} \sin(\phi/2)v(y), \quad (61i)$$

where

$$v(y) = \pi i k \cos \alpha_0 \exp(-ik\xi_0) \frac{\partial}{\partial z_0} \int_{-\infty}^{\xi_0} (r'_0)^{1/2} \sin(\phi'_0/2) \frac{H_1^{(2)}(k\rho')}{\rho'} \Big|_{r=0} \exp(ik\xi'_0) d\xi'_0.$$

The leading terms in the expansions of $\Psi^{(2)}$, $\Psi^{(3)}$ are:

$$\Psi^{(2)} \sim r^{-1/2} \sin(\phi/2)v(y) \tan \alpha_0 \quad (61ii)$$

$$\Psi^{(3)} \sim -r^{-1/2} \cos(\phi/2)v(y) \sec \alpha_0. \quad (61iii)$$

For the net edge behavior of the electric field components, we see from (58) that we must add terms arising from the differentiation of Φ to the $\Psi^{(1)}$, $\Psi^{(2)}$, $\Psi^{(3)}$ terms. The terms to be added are obtained from the expansion

$$\Phi \simeq 2\pi(rr_0)^{1/2} \sin(\phi/2)(\sin \phi_0/2)(\rho_0)^{-1} H_1^{(2)}(k\rho_0)$$

($\rho_0 = \rho|_{r=0}$), which follows from our expression for F [see (50)] upon our noticing that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_0}\right)F = -\frac{M}{k} = \frac{\pi(r+r_0) \sin(\phi/2) \sin(\phi_0/2)}{(rr_0)^{1/2}} \frac{H_1^{(2)}(k\rho)}{\rho}.$$

The explicit results for the leading terms are

$$e_\xi \sim r^{-1/2} \sin(\phi/2) \left(\frac{i}{k} \frac{\partial v}{\partial \xi_0} \right), \quad (62)$$

$$e_\eta \sim r^{-1/2} \sin(\phi/2) \left(\frac{i \tan \alpha_0}{k} \frac{\partial v}{\partial \xi_0} \right), \quad (63)$$

$$e_z \sim r^{-1/2} \cos(\phi/2) \left(\frac{\sec \alpha_0}{ik} \frac{\partial v}{\partial \xi_0} \right), \quad (64)$$

where $v(y)$ is given above under (61i).

To complete our description of the field near the edge of the screen, we consider (47), (48), (49). The results for the magnetic field components as $r \rightarrow 0$ are found to be

$$h_\xi = 0, \quad (65)$$

$$h_\eta \sim (r)^{1/2} \cos(\phi/2) \cdot 2i(\epsilon/\mu)^{1/2} \sec^2 \alpha_0 \frac{\partial v}{\partial \xi_0}, \quad (66)$$

$$h_z \sim (r)^{1/2} \sin(\phi/2) \cdot 2(\epsilon/\mu)^{1/2} \tan \alpha_0 \sec \alpha_0 \frac{\partial v}{\partial \xi_0}. \quad (67)$$

It is evident from (66) that our edge condition (8) is satisfied. We remark also that the edge behavior given by (62), (63), (64) verifies that obtained by Toraldo di Francia

[1] in an approximate treatment of diffraction by a unidirectionally conducting small circular disc. The field behavior in the vicinity of a diffracting edge is, as anticipated, independent of the shape of the screen.

5. Summary. The results of the foregoing analysis may be summarized in the form of a theorem:

Theorem. Let a dipole field \mathbf{e}_0 , \mathbf{h}_0 derived from the Hertz vector

$$(0, 0, \Pi_z), \quad \text{with} \quad \Pi_z = (e^{-ikR}/kR),$$

be incident upon a screen

$$x \geq 0, \quad -\infty < y < \infty, \quad z = 0$$

which has infinite conductivity in the ξ -direction, where

$$\xi = x \cos \alpha_0 - y \sin \alpha_0 \quad (0 < \alpha_0 < \pi/2),$$

and is perfectly insulating in the direction η normal to ξ :

$$\eta = x \sin \alpha_0 + y \cos \alpha_0.$$

If the resulting scattered field \mathbf{e} , \mathbf{h} is required to satisfy Maxwell's equations, to be outgoing at infinity, and to meet the conditions (boundary condition, two jump conditions and an edge condition) of unidirectionality:

$$BC: \quad e_\xi = 0, \quad \text{on the screen}$$

$$JC^1: \quad [h_\xi] = 0, \quad \text{across the screen}$$

$$JC^2: \quad [e_\eta] = 0, \quad \text{across the screen}$$

$$EC: \quad [h_\eta] = 0, \quad \text{at the edge of the screen,}$$

and also to be integrable at the edge of the screen, then the total (vector) field

$$\mathcal{E} = \mathbf{e} + \mathbf{e}_0, \quad \mathcal{H} = \mathbf{h} + \mathbf{h}_0$$

is given uniquely by

$$\mathcal{E} = \{\Phi_{\xi z_0} + \Psi^{(1)}, \Phi_{\eta z_0} + \Psi^{(2)}, \Phi_{zz_0} + k^2 \Pi_z + \Psi^{(3)}\},$$

$$\mathcal{H} = i\omega\epsilon \left\{ \frac{\partial \Pi_z}{\partial \eta}, -\frac{\partial \Pi_z}{\partial \xi} + b^{(\eta)}, b^{(z)} \right\},$$

where the components of the vectors \mathcal{E} , \mathcal{H} are in the ξ -, η - and z -directions, and where Φ is Macdonald's classical solution, vanishing on the screen, and given by

$$\Phi = I_R - I_S,$$

with

$$I_R = \frac{i}{2} \int_{-\mu_R}^{\infty} H_1^{(2)}(kR \cosh \mu) d\mu,$$

$$I_S = \frac{i}{2} \int_{-\mu_S}^{\infty} H_1^{(2)}(kS \cosh \mu) d\mu,$$

and

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2},$$

$$S = [(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{1/2},$$

$$\mu_R = \sinh^{-1} \{ (2/R)(rr_0)^{1/2} \cos [(\phi - \phi_0)/2] \},$$

$$\mu_S = \sinh^{-1} \{ (2/S)(rr_0)^{1/2} \cos [(\phi + \phi_0)/2] \},$$

$$x = r \cos \phi, \quad z = r \sin \phi, \quad x_0 = r_0 \cos \phi_0, \quad z_0 = r_0 \sin \phi_0.$$

The functions $\Psi^{(1)}$, $\Psi^{(2)}$, $\Psi^{(3)}$ are defined by

$$\begin{aligned} \Psi^{(1)} = r^{-1/2} \sin(\phi/2) & \left\{ \pi i k \cos \alpha_0 \sin(\phi_0/2) \exp(-ik\xi_0) \right. \\ & \left. \cdot \frac{\partial}{\partial z_0} \int_{-\infty}^{\xi_0} \frac{(r+r')}{(r'_0)^{1/2}} \frac{H_1^{(2)}(k\rho')}{\rho'} \exp(ik\xi'_0) d\xi'_0 \right\}, \\ \Psi^{(2)} = e^{-ik\xi} \int_{-\infty}^{\xi} \exp(ik\xi') & \frac{\partial \Psi^{(1)}}{\partial \eta} \bigg|_{\xi=\xi'} d\xi', \\ \Psi^{(3)} = e^{-ik\xi} \int_{-\infty}^{\xi} \exp(ik\xi') & \frac{\partial \Psi^{(1)}}{\partial z} \bigg|_{\xi=\xi'} d\xi', \end{aligned}$$

where $\rho = [(y - y_0)^2 + (r + r_0)^2]^{1/2}$, and the functions $b^{(\eta)}$, $b^{(s)}$ are given by

$$\begin{aligned} b^{(\eta)} &= \int_{-\infty}^{\xi} (F_{zz} + \Psi^{(3)}) \bigg|_{\xi=\xi'} d\xi' \\ b^{(s)} &= - \int_{-\infty}^{\xi} (F_{\eta z} + \Psi^{(2)}) \bigg|_{\xi=\xi'} d\xi' \end{aligned}$$

with $F = -\Pi_s + \Phi$.

Some of the field components are singular as the edge of the screen is approached, but these singularities are of the physically admissible 0 ($r^{-1/2}$) type, as may be seen from Eqs. (62) through (67) above. It is especially to be noticed, among our results on edge behavior, that the y -component of the electric field vanishes along the edge of the screen; this follows either from (62), (63) or from (38). We conclude that a suitable edge condition for our problem is

$$e_y = 0,$$

at the edge of the screen, exactly as for a perfectly conducting screen of the same geometry. The use of the corresponding condition, namely that the field component tangential to the rim of the screen be required to vanish, is therefore suggested for further investigations of unidirectional screens. This condition is met by the approximate solution of Toraldo di Francia [1].

We remark that we have given the highest order terms in r , in discussing edge behavior: it is assumed that r_0 is finite. If however we let $r_0 \rightarrow \infty$, the character of the results for field behavior near the edge simplifies further. This type of approximation was given by Senior [4] in discussing edge behavior for the incidence of a dipole field on a perfectly conducting screen.

We conclude by verifying our theorem. Let us write Maxwell's equations in the form

$$(\Delta_{\xi\eta z} + k^2)\mathcal{E} = 0,$$

$$\nabla \cdot \mathcal{E} = 0,$$

$$i\omega\epsilon\mathcal{E} = \nabla \times \mathcal{H}.$$

Notice now that $\Psi^{(1)}$ is a wave function, and that $\Psi^{(2)}, \Psi^{(3)}$ are therefore wave functions. It is then clear that $b^{(\eta)}, b^{(z)}$ are wave functions. Remarking [see (55), (59)] that

$$\frac{\partial \Psi^{(1)}}{\partial \xi} + \frac{\partial \Psi^{(2)}}{\partial \eta} + \frac{\partial \Psi^{(3)}}{\partial z} = k^2 F_z.$$

we find that $(\mathcal{E}, \mathcal{H})$ is an electromagnetic field. The boundary condition

$$e_{\xi} = -e_{0\xi}$$

is satisfied; and the first jump condition is obviously met, since $h_{\xi} = 0$. We have pointed out in giving $\Psi^{(2)}$ that it is an even function of z ; and it is clear that e_{η} is continuous across the screen, since $\Phi_{\eta z}$ is. The edge condition $[h_{\eta}] = 0$ at the edge of the screen follows from the edge behavior of h_{η} .

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