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BOOK REVIEWS

Lineare algebra. By Werner Graeub. Springer-Verlag, Berlin, Gottingen, Heidelberg, 1958. x + 219 pp. $9.35.

A remarkably complete text on the subject of the title. By linear algebra is meant the theory of finite dimensional vector spaces and the linear and multilinear functions on them. This theory is conveniently subdivided into three parts, the general theory which applies to vector spaces over an arbitrary field, the theory of real vector spaces, and the theory of complex vector spaces. The principal results in all three branches are presented here. The book is divided into eleven chapters. The first two on "Lineare Räume" and "Lineare Abbildungen und Gleichungssysteme" are prerequisites for all the chapters which follow. Most of these latter, however, can be read independently of each other. Chapter III "Determinanten" is needed in order to read Chapter IV "Orientierte Lineare Räume" which deals with real vector spaces and is perhaps the most unusual part of the book. The main result of the chapter states that the basis x_ν can be deformed continuously into the basis y_ν if, and only if, the determinant of the mapping $x_\nu \to y_\nu$ is positive. Chapter V "Multilineare Algebra" is the longest in the book and treats the algebraic theory of tensors in an invarient (coordinate free) manner à la Bourbaki. The next four chapters are concerned with real vector spaces. Chapter VI "Der Euklidische Raum" introduces the scalar product. Chapter VII "Lineare Abbildung Euklidischer Räume" is concerned with Eigen values and the spectral theorem for self adjoint transformations. Chapter VIII "Symmetrische Bilinearfunktionen" treats quadratic forms and the inertia index. Chapter IX "Flächen zweiter Ordnung" presents both the affine and Euclidean classification of quadric surfaces. Chapter X "Unitäre Räume" deals with the spectral theorem for Hermitian and normal transformations. The final chapter XI "Invariante Unterräume" returns to the general theory giving an invarient treatment of canonical forms of transformation and the Cayley-Hamilton Theorem.

The book is suitable for a text, probably on the graduate level, and has many exercises in each chapter. I suspect it will be even more useful as a reference book in the subject for practicing mathematicians who are not specialists in this particular field.

DAVID GALE

Readings in linear programming. By S. Vajda. John Wiley & Sons, New York, 1958. vii + 99 pp. $3.00.

In this introduction to linear programming typical applications are illustrated by numerical examples which suggest the appropriate concepts and techniques. Only very elementary algebra is required for the understanding of the book, and the exposition is clear and easy to follow. The contents of the book are indicated by the following (incomplete) list of chapter headings: Transportation Problem, Caterer Problem, Production Scheduling, Transhipment, Bid Evaluation, Flow through a Network, Ship Scheduling, Personnel Assignment, Routing Aircraft, Investment, The Simplex Tableau, Nutrition Problem, Airlift, Blending of Aviation Gasolines, Smooth Patterns of Production, Duality, Selection of Products, Train Loss Reduction, Attendant's Rota, Warehousing, Games, Bibliography.

W. PRAGER

Calculus of variations and its applications. L. M. Graves (Editor). McGraw-Hill Book Company, Inc., New York, 1958. v + 153 pp. $7.50.

This important book contains the papers which were presented at the Eighth Symposium in Applied Mathematics sponsored by the American Mathematical Society and the Office of Ordnance Research, and held in April, 1956. The papers are noteworthy for their clarity of exposition and avoidance of an excessively cramped style, so that they should be of interest to physicists, engineers and operations analysts, as well as to mathematicians. They cover a wide range of topics in the calculus of variations and its applications to elasticity, plasticity, electromagnetic theory, mathematical economics and other fields, as the following list of authors and titles shows:

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BOOK REVIEWS

(*Continued from p. 254*)

1. E. Reissner, "On Variational Principles in Elasticity"
1. D. C. Drucker, "Variational Principles in the Mathematical Theory of Plasticity"
3. P. G. Hodge, Jr., "Discussion of D. C. Drucker's Paper 'Variational Principles in the Mathematical Theory of Plasticity' "
4. J. B. Keller, "A Geometric Theory of Diffraction"
5. J. B. Diaz, "Upper and Lower Bounds for Eigenvalues"
6. J. L. Synge, "Stationary Principles for Forced Vibrations in Elasticity and Electromagnetism"
7. H. F. Weinberger, "A Variational Computation Method for Forced-Vibration Problems"
8. M. M. Schiffer, "Applications of Variational Methods in the Theory of Conformal Mapping"
9. R. Bellman, "Dynamic Programming and Its Application to Variational Problems in Mathematical Economics"
10. S. Chandrasekhar, "Variational Methods in Hydrodynamics"
11. E. H. Rothe, "Some Applications of Functional Analysis to the Calculus of Variations"

Unfortunately, only a one-page index is provided.

The personal tastes of the reviewer governed the choice of three of the longer papers for further discussion.

Keller introduces a generalized theory of geometrical optics designed to include the effects of diffraction. This is accomplished through the introduction of new rays, diffracted rays, which account for the appearance of light in shadow regions and which alter the light found in illuminated regions. These diffracted rays arise when a ray impinges on an edge or a vertex, or grazes an interface. First the diffracted rays are specified explicitly by describing the rays which arise under various circumstances. Then the rays are characterized by an extension of Fermat's principle, with the equivalence of the two descriptions following from standard variational considerations. Following this diffracted wavefronts, eiconal functions, and imaginary rays (which account for the light found in the region on that side of a caustic through which no ordinary rays pass) are introduced. Finally, to achieve a quantitative theory, amplitude and phase functions are associated with the rays. The author states, justifiably, "if this ray theory had been available at the time of the controversy between ray and wave theory, it might have forestalled the acceptance of the latter." The paper concludes with an interesting section relating this theory to earlier works in diffraction theory, along with an extensive bibliography.

The determination of approximate values for the characteristic numbers of self-adjoint differential operators is of great practical importance. The Rayleigh-Ritz method furnishes upper bounds and the method of Weinstein lower bounds. Using the equation for the vibrating clamped plate as an example, Diaz states Courant's minimax characterization of the eigenvalues and notes that by contracting the class of admissible functions upper bounds are obtained, and by enlarging it, lower bounds, which leads to a unified view of the basic ideas underlying these two methods of approximation. Then each is discussed in detail, along with Aronszajn's extensions. The paper is rounded out with some remarks concerning estimates of the errors and relations between the eigenvalues for the vibrating plate and vibrating membrane problems. Though no numerical examples are given, many references to the literature are provided which still further enhance the value of the paper.

The purpose of Bellman's paper is the discussion of a variety of optimization problems arising in mathematical economics. The approach is via the functional equation technique of dynamic programming, a field in which the author has pioneered and which has been extensively explored in recent years. Interest centers on the analytical and computational determination of optimal policies for use in multistage decision processes. The key to the solution of these problems is provided by the principle of optimality: an optimal policy has the property that whatever the initial state of the system is, and whatever the state is that results from the initial decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the initial decision. In turn, this is but a special case of the more general principle of invariant imbedding (*Proc. Nat. Acad. Sci.*, v. 42 (1956), p. 629), which can be used in treating a number of problems in mathematical physics. Following a brief outline of dynamic programming and the relationships between continuous decision processes and the calculus of variations, several multi-stage allocation processes and smoothing processes (of both deterministic

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(*Continued from p. 262*)

and stochastic type) are discussed in some detail. The paper closes with the solution of a problem arising from consideration of a miniature mathematical model of the interdependent steel and automobile industries. This leads to a bottleneck problem involving the allocating of steel resources over a given period of time to the production of (a) additional steel itself, (b) additional steel-producing facilities, and (c) automobiles, where the objective is to maximize the total number of automobiles produced during the process.

Even a casual perusal of this volume will impress the reader with the role of the omnipresent variational principle in applied mathematics.

ROBERT KALABA

Boundary layer effects in aerodynamics. International Symposium held at National Physical Laboratory, England, in 1955. Philosophical Library, New York, 1957. $12.00.

This volume contains a complete record of an International Symposium held at the National Physical Laboratory, England, in the Spring of 1955. Nine papers were presented, covering most aspects of the subject of current interest. The Symposium was opened by Professor L. Howarth, with a broad and enlightening survey of boundary layer theory as it stood in 1955.

The first two papers are essentially of a theoretical character. A paper on three dimensional boundary layers by Timman analyses the flow on a general surface in terms of intrinsic coordinates with application to thin wings and yawed elliptic cylinders. Lighthill and Glauert investigate laminar boundary layer flow on a long thin cylinder and obtain solutions valid near the nose and a long way downstream. In a paper on the stability of boundary layer flow on a rotating disk experimental work by Gregory and Walker is compared with theoretical analysis by Stuart. In many respects the two approaches show good agreement and discrepancies which do arise are satisfactorily explained.

An analysis of boundary layer transition by Schubauer and Klebanoff lends strong support to the turbulence spot theory of Emmons. Küchemann considers viscosity effects on swept-back wings with emphasis on separation and the growth of tip vortex sheets. Pankhurst describes various methods of boundary layer control including the use of suction on thick wings.

Young and Kirkby calculate the profile drag on supersonic wings of biconvex and double wedge sections. The symposium ends with two long papers on Shock Wave Boundary Layer Interaction. Holder and Gadd review the work done on this problem on the flat plate and estimate the influence of the effect on the calculation of base drag. The second paper by Pearcey is concerned with Turbulent Boundary Layers on Transonic Airfoils.

Most of these papers have already been published separately but the collected proceedings also contain accounts of the discussion following each paper. This is invariably interesting and in many cases, quite extended.

M. HOLT

Integral equations and their applications to certain problems in mechanics, mathematical physics and technology. By S. G. Mikhlin. Pergamon Press, New York, London, Paris, Los Angeles, 1957. xii + 338 pp. $12.50.

This is a useful addition to the literature in English on integral equations and their applications to continuum mechanics. The first part of the book, which comprises approximately two-fifths of the whole, provides a readable account of the fundamental theory of integral equations. The classical theory due to Fredholm and the Hilbert-Schmidt theory for equations with symmetric kernels are treated in the first two chapters, and the third chapter is devoted to singular integral equations. Methods for approximate solution of the equations are given together with numerical examples. For the most part detailed proofs are given for the theorems used to develop the theory.

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The second part of the book contains a variety of applications, almost all of which are two-dimensional problems. As is to be expected and desired in a work of this nature, the emphasis is on the method of application of integral equations to obtain the solution, so that the working is carried only to the point where detailed results can be found by straightforward calculations. Discussions are included of the integral equation approach to the solution of the wave equation and the heat conduction equation but most of the examples are taken from Russian work in the plane theory of elasticity and hydrodynamics.

The translator deserves thanks for a translation which very seldom jars. However, the reader may be somewhat startled to find that Hooke has lost his final "e" due to the strain of being re-transliterated into English, and one may seek in vain for references numbered higher than thirty-five.

R. T. SHIELD

Vibration and impact. By Ralph Burton. Addison-Wesley Publishing Co., Reading, Mass. x + 310 pp. $8.50.

The author states that this book is intended for senior undergraduate, and for beginning graduate students, and the objectives of the text are listed as treatments of (1) "the natural frequency" (free vibrations without damping), (2) "forces which tend to suppress vibrations" (free vibrations with damping), (3) "periodic forced and transient forced vibrations", and (4) "self-excited vibrations". There are chapters on free vibration, vibratory systems commonly found in machinery, damping, impact, nonlinear vibrations, measuring instruments and analogs, numerical computation of natural frequencies (of multi-modal systems), waves, vibrating beams and related subjects, (the related subjects including such items as plate and membrane vibrations, and damping in a system having two lumped masses, two springs and one viscous damper), and chapters on the analysis of control systems, and fatigue. Nearly every chapter contains some examples and is followed by a number of problems.

In view of the title of the text, the reader may expect to find a book devoted in equal parts to vibrations and to impact; he will be disappointed. There is one chapter of approximately twenty pages dealing with elementary aspects of the response of rigid bodies, elastically supported, to non-periodic, time-dependent forces. In another chapter, there is a short qualitative discussion of wave propagation, wave interaction, and wave reflection from discontinuities, consisting of largely verbal results deduced from heuristic arguments; however, this latter material does not add greatly to the study of impact on elastic bodies.

The instructor who intends to use this book as a text in senior undergraduate, or in graduate, courses must expect students who have no background whatever in ordinary differential equations but a fair preparation in partial differential equations, for there are quite lengthy derivations of the solutions of linear, homogeneous equations with constant coefficients, while the solution to the one-dimensional wave equation (the only partial differential equation in the book) is given without any derivation whatever.

To the reviewer, the book appears disconnected, badly arranged and replete with errors of commission and omission. To list some of the former, the author says of the equation $-kx = m\ddot{x}$ that "unfortunately, we cannot solve this equation by direct integration". The "perturbation method" is illustrated by the example of the pendulum, and the author believes that this method consists of annihilating the nonlinear terms by amputation; thus the "perturbation solution" of $\ddot{\theta} + (g/l) \sin \theta = 0$ is obtained by solving $\ddot{\theta} + (g/l)\,\theta = 0$. The author notes that "Fourier analysis cannot be applied to the solution of nonlinear systems under periodic nonharmonic forcing. . . . Any solution based upon superposition of solutions is invalid . . .". The criterion for stability of vibrating systems is said to be the same as that for static stability—i.e., that the system ". . . should return to its initial configuration after a small disturbance". Of Raleigh's method, it is said that "the more boundary conditions the assumed solution meets, the better will be the estimated frequency". It is also said that "It can be shown that

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estimated first mode frequencies [by the Raleigh method] will ordinarily be too high". One could add many more examples to this list of incorrect or misleading statements.

In some respect, the book lacks order. Aside from questions of sequence of subject matter which appears to the reviewer often unmotivated, one finds a portion of a discussion of the nonlinear pendulum problem, as well as that of cases of nonlinear damping, in Chapter 2 and 4, while nonlinear vibrations are discussed in Chapter 7. In Chapter 12, dealing with vibrating beams and related subjects, one finds that "mention of damped vibration with many degrees of freedom has been avoided up to this point . . ." while the dynamic vibration absorber with damping, and the Lanchester damper are treated in Chapter 9.

Perhaps more serious than the errors of commission are those of omission. One of these is the virtual absence of references. Instead, each chapter is followed by a list of "suggested reading". These lists show a serious disregard, both for important work done in the field, and for the relation of the elementary level of the book under review to the advanced level of some of the suggested readings. As one example for each may serve (1) the omission of McLachlan's book on "Ordinary non-linear differential equations" in the section on nonlinear vibrations, and (2) the inclusion of Mindlin's paper on the "Influence of rotary inertia and shear on flexural motions of isotropic elastic plates". This latter is suggested reading for students from whom even the elementary equation of motion of the one-dimensional vibrating beam was withheld in the body of the book, and who have been left completely in the dark on the existence of rotary inertia and transverse shear effects in beams.

No mention is made of Lagrange's equations of motion or of generalized coordinates; in fact, there is no treatment of theoretical mechanics. The section on nonlinear vibrations contains no phase-plane considerations, no perturbation method, no iteration method and, not a single differential equation. Duffing's name or equation are not mentioned and, although there is a paragraph on relaxation oscillations, Van der Pol's name or equation are also overlooked. The section on vibrating beams, plates and membranes (in that order) fails to derive or even to state the equations of motion of any of these systems. In the treatment of plates, Rayleigh's method is applied to an approximate energy expression without stated restrictions on plate thickness or vibrational amplitudes. In a discussion of the limitations of beam theory as given in the book, reference is made to "massive extensions . . . attached to the beam", and to gyroscopic action, but none to rotary inertia or transverse shear. A chapter on control systems contains neither operational methods nor transfer functions, and the suggested reading includes "Weiner, Norbert, Cybernetics". A final chapter on fatigue presents some largely descriptive paragraphs; phenomenological or solid state physics theories and statistical approaches are ignored.

R. M. ROSENBERG

Handbook of supersonic aerodynamics. Volumes 1, 2, 3, 4, 5. Supt. of Documents, U. S. Government Printing Office, Washington 25, D. C.

The purpose of this Handbook is to provide a compilation of formulae, tables and other related information which would be of use to 'designers of supersonic vehicles.' The volumes are issued in loose-leaf form and all are not as yet complete, provision having been made for the insertion of additional material as it becomes available. When completed, this series will consist of six volumes containing 21 sections. Some idea of their content may be gathered from the following list of titles of sections which have thus far appeared:

Vol. 1: Section 1—Symbols and nomenclature; Section 2—Fundamental equations and formulae; Section 3—General atmospheric data; Section 4—The mechanics and thermodynamics of steady one-dimensional gas flow. Vol. 2: Section 5—Compressible flow tables and graphs. Vol. 3: Section 6—Two-dimensional airfoils. Vol. 4: Section 12—Aeroelastic phenomena. Vol. 5: Section 15—Properties of gases.

W. H. REID

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BOOK REVIEWS

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Some aspects of the mathematical theory of control processes. By R. E. Bellman, I. Glicksberg, O. A. Gross. The RAND Corp., Santa Monica, California, 1958. xix + 244 pp.

The purpose of the book is to provide "a taste of the mathematical theory of control processes", and this has been accomplished by a blending of techniques and considerable mixing of a variety of topics. The authors selected the ingredients from their own contributions while at the same time giving references to more complete discussions and to the work of others. The book is far superior to and much more interesting than a survey. The authors have purposely not gone deeply into the mathematical theory nor have they emphasized applications. The reader interested in either of these aspects can consult the references, and the authors so advise him.

The most interesting feature of the book is the emphasis placed on the formulation of control problems and the techniques available for their solution and the illustration that by another formulation of the same problem new techniques become applicable. The techniques are those of differential equations, variational calculus, linear (Hilbert) spaces, dynamic programming and game theory.

A fairly general description of the control problem—sufficiently general to encompass problems in control of interest to analysts and operations researchers, economists, management consultants, and engineers—is given. The state of a physical system is described at time t by an n-dimensional vector

$$x(t) = (x_1(t), x_2(t), \cdots, x_n(t)).$$

The rate of change of the state of the system is described bv an ordinary differential equation

$$\frac{dx}{dt} = G(x(t), f(t)), \qquad x(0) = c,$$

where the freedom in the choice of the control function f represents our ability to control the system, and we wish to improve the performance of the system by a propitious choice of f. The performance of the system under control f is measured by a functional $J(f)$, and an optimal choice of the control function f is that which minimizes (or maximizes) $J(f)$.

Although it is impossible here to describe all of the various control problems discussed in this book, we can describe the three principal problems. In each of these the differential equation is assumed to be linear with constant coefficients—

$$\frac{dx}{dt} = Ax + f, \qquad x(0) = c.$$

A is a constant matrix—and the control enters as a forcing term. The state x of the system (the output) is then linearly related to the control f (the input) by

$$x(t) = y(t) + \int_0^t K(t - s)f(s)\, ds,$$

where y is the solution of the homogeneous equation (the uncontrolled system). The first control problem is one where the functional J and the constraints on the allowable control functions are linear. For instance,

$$J(f) = \int_0^T (a, x)\, dt\,,$$

$$0 \leqq f_i(t) \leqq m_i\,, \qquad 0 \leqq t \leqq T, \qquad i = 1, \cdots, n,$$

and

$$\int_0^T (f, b)\, dt \leqq k\,,$$

where

$$(y, z) = \sum_{i=1}^{n} y_i z_i$$

is the inner product.

In the second control problem the functional is nonlinear and there are no constraints. An example of a nonlinear (quadratic) functional is

$$J(f) = \int_0^T (x - c, x - c)\, dt + a \int_0^T (f, f)\, dt.$$

This can be interpreted as a desire to maintain the system in the state c during the time interval $[0,T]$. The first term represents the cost of deviation from the state c, and the second term measures the cost of control. Optimal control minimizes the cost $J(f)$. The third control problem is one in which the functional J is nonlinear and there are linear constraints.

Part I of the book is a survey of fundamental results on linear functional equations (primarily, difference and differential equations). Part II is concerned with control problems of the first and second kind. A theoretical solution of a problem of the first kind is obtained using the Neyman-Pearson lemma. The techniques involved in computing the optimal control functions are illustrated. Problems of the second kind are solved by Hilbert-space techniques. Here again computation of a solution is discussed. Part III is a study of three problems of the third kind each attacked differently. First a variant of the Neyman-Pearson lemma is used, and next a problem with constraints is solved by a combination of classical techniques and *ad hoc* methods. The third problem is the "bang-bang" control problem and is solved by methods which utilize techniques of classical differential equations, linear spaces and dynamic programming. Their solution of the problem provides information on how to determine optimal switching in a bang-bang system, and this is illustrated for a two-dimensional system. In this example we see an aspect of the control problem that this book neglects. The practical problem—and this is certainly true of a servomechanism—is to determine the control function, not as a function of time, but as a function of the state of the system. In this case their methods give such a solution.

In Part IV there is an introduction to the theory of dynamic programming which gives a general formulation of both deterministic and stochastic multistage decision processes. The variational problem with constraints, which was previously treated by classical techniques, is now considered to be a multistage decision process. The discrete version of the continuous process provides a means of studying the structure of the solution and gives a numerical method of computation. Part V corresponds to Part IV with game theory in place of dynamic programming. The min-max technique of game theory is applied to some control problems with nonanalytic functionals J.

The authors provide, as they say, a taste of some aspects of control theory in which they have specialized and do so remarkably well. The book contains some misprints, some errors and a few false statements, none of which should cause the intelligent reader great concern. They have covered a wide variety of topics in a vast and exciting field of research with great skill.

J. P. LaSalle

Mathematical theory of compressible fluid flow. By Richard von Mises. Completed by Hilda Geiringer and G. S. S. Ludford. Academic Press, Inc., New York, 1958. xiii + 514 pp. $15.00.

Hilda Geiringer (Mrs. R. von Mises) and G. S. S. Ludford have done the scientific community a great service in completing for publication the last work in compressible flow of Professor von Mises. Unfortunately the present text is only the first part of what the author had originally intended to be a comprehensive work on compressible flow. However, this cannot be considered to detract from what has been published, but rather to leave uncovered by Professor von Mises' unique approach various topics of importance in compressible flow. The book is written principally from the applied mathematical point of view and is divided into five chapters, of which the first three were written by the author and the last two were completed according to his plan following his papers and lecture notes.

In the first chapter the momentum and energy equations are derived including the effect of viscous

stresses and heat conduction. Although the concept of a general specifying equation of state is introduced most of the special cases refer to a perfect gas. The chapter concludes with a discussion of the propagation of small disturbances in an inviscid fluid, along with a delineation of subsonic and supersonic motion. The second chapter presents general theorems of fluid motion and the method of characteristics, with principal emphasis on the mathematical aspects of problems in two independent variables.

The third chapter considers one-dimensional flow, with the first section treating the steady case including viscosity and heat conduction, while succeeding sections deal with the nonsteady flow of an ideal fluid. The chapter concludes on a discussion of shock phenomena with the treatment restricted to a perfect gas of constant specific heat ratio. In the fourth chapter the author deals with plane steady potential flow and discusses the hodograph method, simple waves, exact solutions, and limit and branch lines. In the final chapter the author expands his discussion of hodograph techniques, introduces the oblique shock in a perfect gas, and concludes the book with a stimulating discussion on the existence of smooth transonic flows. For the reader who is interested, there are some forty pages of notes and addenda which supply interesting historical footnotes to the text.

Certainly Mrs. von Mises and Professor Lundford are to be congratulated on the manner in which they completed the text, for the reader's immediate impression is that the book was written in its entirety by Professor von Mises. Of course, as in any book as detailed as the present one, it is always possible to disagree with certain features of the presentation. Thus, this reviewer felt at times a lack of Professor von Mises' apt physical description of fluid flow phenomena. This is particularly manifest in the presentation of characteristics and in the later discussion of simple waves. In addition, this reviewer did not always find a consistent level of treatment as evidenced, for example, by the unecessary restriction to a perfect gas in treating shock waves directly after introducing a general state equation. Some difficulty may also be encountered as a result of the manner in which some of the material is presented, an example of which is the introduction of one-dimensional shock structure before any consideration is given to the inviscid Hugoniot shock conditions.

In spite of these minor criticisms, to the reader who has some acquaintance with the field of compressible fluid flow and who desires a text which presents the more mathematical aspects of the subject, as well as to students and research workers more directly concerned with compressible flow as considered from a mathematical point of view, the book is highly recommended. In the years to come this book is certain to be recognized for its fund of information. It can be considered a fitting tribute to the author.

RONALD F. PROBSTEIN

Einige nichtlineare Probleme aus der Theorie der selbsttätigen Regelung. By A. I. Lurje. Akademie-Verlag, Berlin, 1957. xi + 167 pp. $3.60.

This monograph represents a revised and enlarged edition of papers written in the years 1945–1950. Its purpose is to bring the modern theory of nonlinear control to the engineers designing such control systems. Therefore, after careful exposition of methods and solutions, particular examples are treated in each chapter.

The book consists of four chapters: The canonical form of the equations in the theory of automatic control; the stability of control systems with one controlling element; self oscillations in control systems; behavior of a control system at the boundary of the stability region. The first chapter serves as a base for the following ones, which are independent of each other.

The state of the controlled system is described by n variables φ_k. In the direct control their deviation $\Delta\varphi_k$ from the desired equilibrium values determines the action ξ of a controlling element (e.g. motor) which should reestablish the equilibrium of the controlled system. In case of feedback a linear combination (called σ) of ξ, its derivatives and the $\Delta\varphi_k$ determines the action of the motor. The differential equations describing the state of the controlled system are linear; the nonlinearity enters through the characteristic of the controlling motor: $\dot{\xi} = f(\sigma)$. Various realistic functions $f(\sigma)$ are considered in the examples.

The canonical form of the differential equations describing the state of the controlled system is derived without the use of matrices, but with ample use of determinants and Dirac's delta function.

In Chapter II the stability in the large of the control system is investigated with as little restriction as possible concerning $f(\sigma)$. A Liapounoff function is constructed for the problem in question with the assumption that $\sigma f(\sigma) > 0$. The conditions for stability of a given system are determined by a system

of n quadratic equations, whose solution for $n > 2$ is tedious and for $n > 5$ extremely difficult (Lurje's own statement). In addition the special case $f(\sigma) = c\sigma + \Phi(\sigma)$ with $c = \text{const.} > 0$ and $\sigma\Phi(\sigma) > 0$ or $\sigma\Phi(\sigma) < 0$ is investigated.

In the third chapter Lurje studies the occurrence and stability of self oscillations in nonlinear systems by the method of Kryloff and Bogoliuboff and by the method of Poincaré. Both methods are explained in detail, based on a paper by Bulgakoff. Readers principally familiar with the methods will enjoy the discussion of their special merits and difficulties in the present problem.

The fourth and last chapter of the book is based on a paper by Bautin which in turn used a method suggested by Liapounoff in 1935. (Unfortunately no translation of this paper of Liapounoff seems to be available; his famous book, on "the general problem of stability" (1892) was translated into French; some chapters of the latter are expected to be known to the reader of Chapter II). The boundary of the stability region is considered to consist of dangerous and not dangerous portions. A disturbance starting outside, but close to the not dangerous portion will not deviate too much from a stable motion. On the other hand motions with initial values close to, but outside the dangerous boundary of the stability domain may deviate considerably from the stable motions.

The translation of Lurje's monograph into German is well done and certainly helps getting acquainted with these investigations, so important for the design of control systems.

I. FLÜGGE-LOTZ

Lectures on ordinary differential equations. By Witold Hurewicz. The Technology Press of the Mass. Institute of Technology, Cambridge, John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. xvii + 122 pp. $5.00.

This book is a reprint of lecture notes of a course given by Hurewicz in 1943. It is a very clear and readable introduction to the fundamentals of the theory of systems of differential equations.

The first and second chapters deal with questions of existence and uniqueness of solution, the third deals with the properties of linear systems with constant or variable coefficients, the fourth with the theory of the singularities of second order systems, saddle points, nodes, foci, etc., leading up to a discussion of the periodic solutions of second order nonlinear differential equations.

For those who are interested in following the more modern developments, there is a list of twelve volumes dealing with various aspects of the theory of ordinary differential equations.

RICHARD BELLMAN

High-speed data processing. By C. C. Gotlieb and J. N. P. Hume. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1958. xi + 338 pp. $9.50.

The field of high-speed data processing has become a highly specialised branch of the general field of computing, and there has been a need for an account of its peculiarities, equipment and techniques, although it is still in a state of rapid development. This book goes some way to fill this need, and is particularly suited to the newcomer to the computing field who expects to specialise in data processing, but those already experienced in the use of general-purpose computers will find many of their questions unanswered.

When trying to deal with a restricted aspect of the computing field which depends so much upon new and changing equipment there is a strong temptation to stray into the ramifications of computer design and programming principles and niceties, but some discussion of these aspects is unavoidable in the absence of books dealing with them specifically. Although such preliminaries occupy a little over half the book, an eye is kept on those aspects particularly applicable to data processing, and the effect of the special requirements of data processing of equipment compared with those of scientific computation are discussed in the earliest chapters, and throughout, actual machines are frequently referred to.

The treatment of representation of information concentrates upon alpha-numeric data and coded decimal systems. This considerably assists later discussion on programming and coding. Knowledge of the details of binary arithmetic is not essential and is correctly relegated to an appendix.

Modern data processing systems are generally centered around a 'general-purpose' type of computing and control unit which usually possesses an instruction code specially adapted to the class of work to be carried out; thus there is special emphasis on input, output, store transfers and comparisons and so on. The organization within the central unit depends upon well established principles, and the discussion of the details of the central unit and various storage systems would seem to be excessive compared with the treatment of the various means of input and output upon which data processing so largely depends.

The discussion of instruction code types and address systems is followed by a lengthy section on programming and coding, and the authors have adopted a hypothetical instruction code as a model. This code is readily understood and remembered, being single-address coded-decimal with mnemonic letter code for operation type. It contains special features to assist coding for data processing, including block transfers, input and output buffers, a number of operations to assist double length operations within the accumulator and storage of discriminations. It does however have the disadvantage, for the beginner, of packing two instructions to a word; these are addressable only in pairs, and this complicates the organisation of control transfers and requires the use of redundant 'skip' instructions. It has to be admitted that some machines are just like this, and in the words of the authors; 'This awkward feature of the Hypothetical Machine has been retained to allow the programmer his occupational prerogative of complaining about the instruction code'.

The main techniques of programming are illustrated via a number of simple problems of data processing type and occupies some 60 pages. The use of autocodes of various kinds is becoming common, and this method of coding is described only in the final chapter.

A considerable simplification in the sections on coding would have been achieved, without lack of reality, if this tendency had been accepted, and coding been described in terms of even a simple autocode. The use of symbolic addresses alone would have been of help, to the beginner. The treatment of machine organization could have been shortened and the more interesting features of data reduction dealt with in the later sections dealing with the use of files, sorting, selecting and practical applications, could have received closer treatment, as for instance, on discussion of the various ways of marking file and record endings, the use of additional tracks, and forward and backward reading.

Examples of applications have been taken from the commercial fields including insurance, accounting, planning and scheduling, and there is some discussion of the use of processors as simulators. This latter use would seem to have very great potential for the future development of data processing and would have justified expansion. The authors have in fact kept their feet well on the ground in discussing data processing as it now exists, and there is little reference to its future. Among aspects of the future of data processing which one would like to see discussed are the implications of the treatment of the results of continuously recording scientific experiments, an expanding field of data processing which calls for high processing speeds; the possible effects of increased processing speeds on code systems; the likely effects of new electronic techniques; and so on.

The extension of data processing to wider fields is still largely dependent upon the elimination of the human being at the early stages, and the presentation of large amounts of treated information in readily assimilable forms. Future developments would seem to depend upon the design of special input and output devices for translating original data, as occur in banking and national census and market research, into directly usable media without human intervention; and high speed multi-curve plotters for ready assimilation by the reader.

The book makes a good introduction to the current problems and techniques of data processing, it is excellently clear in style, and contains, not least in importance, an extensive bibliography.

T. PEARCEY

An introduction to combinatorial analysis. By John Riordan. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. x 244 pp. $8.50

The author interprets combinatorial as "anything enumerative" and presents a fairly systematic survey of the subject with particular emphasis on the developments of the last fifty years and on the use of generating functions. The book should be of interest not only to specialists in the field but should serve as a useful reference as well since, between the text and the problems (of which there are about 200), the book contains the solutions to a considerable number of problems. In addition to the classic

material on permutations, combinations, etc., there is a chapter entitled "Partitions, Compositions, Trees, and Networks" plus two on "Permutations with Restricted Position" much of the material in which is either new or of recent origin.

Although the reviewer could find no specific criticism to make of the book, he did not find it very inspiring. This may be more a consequence of the reviewer's attitude toward the subject than the authors presentation although the authors style of writing does not seem to convey much enthusiasm.

G. F. NEWELL

Electronic digital computers. By Franz L. Alt. Academic Press Inc., New York and London, 1958. x 336 pp. $10.00

The book is divided into five parts: 1. Introduction; 2. Automatic digital computers; 3. Coding and programming; 4. Problem analysis; 5. Matching problems and machines. Parts 1 to 3 are standard material but 4 and 5, which make up well over half of the book, present matter not previously collected in book form. In part 4, methods of numerical analysis are classified and discussed in relation to their usefulness and efficiency when used with computers; and in part 5 the impact of computers on scientific and engineering research is demonstrated by examples drawn from many fields. The exposition is clear and concise, and geared to a mathematical level far below that of the other books in this series.

WALTER F. FREIBERGER

Ordinary differential equations. By Wilfred Kaplan. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1958. xv 534. $8.50.

This is an excellent introductory text in the field of differential equations. It is carefully written and carefully planned, with many important illustrative examples, illuminating graphs, and a large number of exercises.

The first part, about three hundred pages, is devoted to a thorough study of linear differential equations with constant coefficients, n-th order equations, and linear systems. For this latter purpose, matrices are introduced and applied. A number of engineering applications are given and there is a very helpful discussion of the interconnection between various engineering terms and mathematical vocabulary. A long chapter, fifty pages, is devoted to the fundamental technique of power series solutions, and a brief chapter, ten pages, perhaps too brief, to numerical solution via difference techniques.

The penultimate chapter contains a very readable introduction to the study of periodic solutions of nonlinear second order differential equations via phase plane analysis. In the final chapter, we find existence, uniqueness, and convergence theorems which the author has wisely postponed to the end of the volume.

The book is heartily recommended for college classes and for those who wish to prepare themselves for the more advanced theory of differential equations and its applications.

RICHARD BELLMAN

Boundary layer research. Edited by H. Görtler. Springer-Verlag, Berlin, Gottingen, and Heidelberg, 1958. xii 411 pp. $16.20.

This book differs notably from the usual symposium proceedings (so often rather aimless and full of ticket papers). A symposium was actually achieved, by felicitous choice of a subject of neither too wide, nor too narrow, a scope. And while geography played clearly a role in the selection of the participants, the remaining freedom of choice was employed to ensure an average level of contributions well above that usual at such international meetings.

The 31 papers and 22 notes range over the whole field of boundary layer research, with the keenest interest discernible, perhaps, in the subjects of stability and separation.

The volume offers more than a formal record of proceedings, due to the inclusion of a part of the impromptu discussions. These make the meeting come partly alive again, and it was reluctance to

miss any of them (and especially the pungent and illuminating remarks offered by some of the English participants), almost as much as reluctance to miss any of the first-class papers, which made this reviewer read through all 411 pages. It must have been fun to be at the meeting, and it is fun to read its record. Apart from which, the book will be needed on the shelves of all those with a more than transitory or platonic interest in boundary layers.

R. E. MEYER

Elementary statistical physics. By C. Kittel. John Wiley & Sons, Inc., New York, and Chapman & Hall Ltd., London, 1958. ix 228 pp. $8.00.

This book is based upon a series of lectures given to beginning graduate students in physics and is written in a style quite similar to the author's earlier book "Introduction to Solid State Physics". Unlike many authors who have tried to write a book that will serve as both a text and a reference but succeed in doing neither, this book is only an elementary text. For the more difficult topics such as ergodic theory, phase transitions, etc., frequently described in books of similar titles, the author simply refers the reader to the more advanced works. The scope, however, is very broad and includes besides the usual statistical mechanics, short sections on irreversible thermodynamics, Brownian motion, noise theory and other topics not always found in an elementary book.

Although the author sacrifices rigor at times, both in the mathematics and in the physics, to preserve simplicity, this seems to be unavoidable in a book of this type. At no place does the author embark on long tedious calculations or deviate very much from a fairly constant level of difficulty. Particularly because of this, the reviewer believes that this is the best elementary text book presently available on this subject. The book is primarily for physicists, however.

G. F. NEWELL

An introduction to multivariate statistical analysis. By T. W. Anderson. John Wiley & Sons, Inc., and Chapman & Hall Ltd., London, 1958. xii 374 pp. $12.50.

This textbook gives a very systematic treatment of multivariate analysis. Throughout most of the book the methods of analysis are derived by the maximum likelihood method (for estimation) and the likelihood ratio criterion (for testing hypotheses). The first of these two methods is known to have certain optimality properties; the second one also has some desirable large sample features and often leads to reasonable tests. In this way the author has organized the material successfully and presents a unified treatment of the many different subjects that belong to multivariate analysis.

After introducing the reader to the multivariate distribution the author describes how to estimate the mean vector and the covariance matrix and how the empirical correlation coefficients are distributed. One chapter is devoted to the T^2-statistic and the Behrens-Fisher problem.

The next chapter deals with a different type of problem, namely the classification of observations into one or two or more normal populations. This topic is discussed from the point of view of statistical decision functions in terms of Bayes procedures, admissible classes, etc.

After a detailed discussion of testing various multivariate hypotheses the author turns to principal components and canonical correlations, which are treated in a very lucid way.

In the last chapter some more advanced problems are sketched briefly, and it is only to be regretted that they were not given more space in the book; this is true especially about factor analysis.

An appendix on matrix theory ends the book. At the end of each chapter a number of problems are given.

This book is an attractive and representative example of the modern point of view on theoretical questions among mathematical statisticians in this country. It should be very useful to anybody interested in multivariate analysis.

ULF GRENANDER

Elasticity and plasticity. By J. N. Goodier and P. G. Hodge, Jr. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. ix + 152 pp. $6.25.

This book is the first volume of a series of surveys in Applied Mathematics. In the preface the authors make a modest statement, saying that their work would be devoted only to such a group of problems in the domain of the theory of elasticity and plasticity which were dealt with in publications rather inaccessible or published in non-familiar languages (non-familiar from the English speaking people's point of view). So it was not the authors' aim to present an exhaustive and proportioned survey of all the branches of the subject in question, but rather to focus attention on those important latest achievements which, according to their opinion, are little known to the readers whose first language is English. What is known and easily accessible has been omitted or only touched upon. Nevertheless, the book plays a much more important role because it represents a kind of monographical approach to some fields of the theory of elasticity and plasticity.

The approach to the subject in Part I (The Mathematical Theory of Elasticity, by J. N. Goodier) is different from that in Part II (The Mathematical Theory of Plasticity, by P. G. Hodge, Jr.), both in essence and form. However, each of them has its own merits, the first part being more a survey, the second a survey with an attempt towards monographical approach.

Part I (47 pages) deals successively with: two-dimensional problems, the problems of holes and fillets (without and with reinforcement), mixed boundary value problems, anisotropic elasticity, thermal problems, three-dimensional contact problems, wave propagation, seismical and vibrational problems. Bibliography contains 128 entries (65 Western, 63 Soviet).

Part II (96 pages) deals successively with the foundations of the theory of perfectly plastic bodies, of strain-hardening bodies, piecewise linear plasticity, the minimum principles, and a number of applications of the theory, such as bending of planes and shells, plane strain and plain stress, beams, rods, and miscellaneous problems. A separate chapter is devoted to Soviet, Polish, Hungarian, and Chinese papers. Bibliography contains: 345 entries (182 Western, 149 Soviet, 11 Polish, 2 Hungarian, 1 Chinese).

The closing part of the book includes Author Index and Subject Index (8 pages).

The book is interesting and refreshing. Both authors, authorities in their specialities, made a worthy contribution, which is very useful and serves the purpose: for experts, the book represents a valuable contemporary informative source (with a detailed list of references); for non-specialists, it may be considered as an easily understandable introduction to the subject.

W. Olszak

Games and decisions. By R. Duncan Luce and Howard Raiffa. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1957. xix + 509 pp. $8.75.

Mathematical fields, as all other subject areas, are in constant need of redefinition. Applied mathematics is no exception to this rule. One of the most interesting phenomena of the past couple of decades is the wide application of mathematical techniques to areas that previously lacked them. The historian of the future may well regard the present era as one of the times when mathematics was most healthily challenged and reinvigorated by the posing of new problems, arising from attempts to apply mathematics to new fields of study.

Among the new areas of application are the nonphysical (or behavioral) sciences such as anthropology, biology, economics, genetics, psychology, management science, and sociology. The book under review gives a wide-ranging discussion of a number of these new kinds of applications, and may properly be regarded as a book on applied mathematics. This is true even though the traditional tools of applied mathematics (such as differential and integral equations) nowhere appear in the book. Instead, the mathematics of convex sets and other kinds of mathematics formerly in the "pure" category are used to state and criticize the results of game theory.

Game theory was put on its mathematical feet in the epic paper of John von Neumann in 1928. This paper was almost unnoticed for 16 years until, in 1944, the treatise *Theory of Games and Economic Behavior,* by von Neumann and the economist Oskar Morgenstern appeared in 1944. There was a brief period of further dormancy of the theory until about 1948 when the first of a flood of papers started appearing on the subject. To show how vast the subject has become in this short time, a bibliography of articles in game theory up to 1957, recently compiled by the reviewer and his wife, lists more than

1000 papers and books on the subject during that period. Some of these articles are expository in nature and were written to introduce the subject to a new audience. However, a substantial fraction of these papers contain the meat of a new result.

It is manifestly impossible for a research mathematician, say, to attempt the reading of this many papers unless he has a very serious professional interest in the subject. But it is completely impossible for a non-mathematician, say an interested behaviorial scientist, to plow through all that material and still maintain himself in his own profession. The authors (both of whom received their principal training in mathematics) have undertaken the task of surveying the field. Because many of the people who want to know these results are not technically trained mathematicians, the authors have merely included the statements and critique of the results together with a bibliography of references for those who wish to probe deeper into the mathematical background.

Although both authors are mathematicians, Luce has concerned himself very much with sociology and Raiffa has made extensive studies of statistics. Both these biases appear in the book. For instance, we find that they talk of a player's "security level" instead of his mathematical expectation—clearly sociological terms. Also the very complete discussion of the statistical decision problem is a reflection of Raiffa's special interests. These emphases are to the good, since they reflect the parts of the theory on which each of the authors has concentrated most fully. It might be noted that the discussion of continuous games is fairly sketchy.

A brief summary of the contents of the book is that it covers the two-person zero-sum game, several of the current solution theories of the n-person non-zero sum game, individual decision making (statistical decisions), and group decision-making (the welfare problem). The content areas to which these subjects are most closely related are sociology, statistics, economics, and management science.

The authors are to be commended for their courage in undertaking such a formidable task as the survey of such a large and recent area of study. The result of their labors is a marvelous exposition of most of the theory. This book is certain to be influential in the future training of students working in the theory as well as in stimulating work on specific research problems. It should be of interest to anyone who wants to find out about new kinds of applied mathematics from either a cultural or technical point of view.

The decision as to whether or not a behavioral scientist or applied mathematician should have this book on his bookshelves is easy—he should.

G. L. THOMPSON

An introduction to fluid dynamics. By G. Temple. Oxford University Press, New York, 1958. xi + 195 pp. $4.00.

The preface states that "the object of this book is to provide an introduction to fluid dynamics, primarily for students reading for honors in mathematics and theoretical physics." One cannot but approve the object and simultaneously acknowledge that the author has achieved it with brilliant success.

Keeping within the bounds of inviscid continuous fluid and with an eye firmly fixed on the physical problem, Temple has produced in 190 pages a fascinating account of hydrodynamics. Particular emphasis has been placed on the "fluid body" that is a portion of fluid which always consists of the same fluid particles. In this connection the argument of 1-3 concerning the application of the laws of motion appears to be unconvincing, for it starts from the tacit assumption that the internal forces form a self-equilibrating system.

The reader is led by easy stages through elementary notions to sources, doublets and vortices, to distributions of these singularities and the action on a body in a uniform stream. Here, in deriving the Kutta Joukowski theorem the author obtains a drag term due to the total source strength in Green's equivalent stratum. To the reviewer this term seems to be necessarily zero, for the normal velocity on the surface of a body at rest in a uniform stream vanishes and it is to this normal velocity that the stratum is due. The text then goes on to conformal mapping, free streamlines, design of wing profiles, axisymmetric flow, and finally slender body theory applied to solids of revolution and checked by exact solutions for the ovary ellipsoid.

The book can be heartily recommended.

L. M. MILNE-THOMSON

Mathematics of physics and modern engineering. By I. S. Sokolnikoff and R. M. Redheffer. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1958. ix + 810 pp. $9.50.

The present volume shares with its well-known predecessor, *Higher Mathematics for Engineers and Physicists*, the aim of providing a discussion of those topics beyond the calculus which are of importance in engineering and physics. Thus, mathematical concepts are first presented in a precise manner and are then illustrated in many cases by examples and problems drawn from engineering and physics, and throughout the book the authors have attempted to preserve this balance between the formal aspects of the subject on the one hand and their application to physical problems on the other. The text is divided into nine chapters which deal with ordinary differential equations, infinite series, functions of several variables, vectors and matrices, vector field theory, partial differential equations, complex variable, probability theory, and numerical analysis, with short appendices on determinants, the Laplace transform, and the Riemann and Lebesgue integrals. Since the chapters are largely independent, the book can easily be adapted to varying teaching requirements and will also prove useful for reference purposes.

W. H. Reid

Introduction a L'Algèbre Supérieure et au Calcul Numérique Algébrique. By L. Derwidue. Masson et Cie, Paris, 1957. 431 pp. $15.80.

The selection of topics in this text book on advanced algebra is especially happy from the point of view of engineers and physicists. The author notes in his preface that, while good texts covering these topics are available in English and German, the literature in French is very sparse. The principal subjects covered are: systems of linear equations and determinants, polynomials, matrices and matrix eigenvalue problems and the stability criteria of Routh, Hurwitz and Schur. In all of these topics great emphasis is placed on bringing the theoretical results down to a form which permits numerical calculation. It is assumed that the reader has access to a desk calculator. Many sample calculations are given indicating appropriate layouts for recording intermediate results and pointing out methods for carrying along "running checks." Ill-conditioned systems are mentioned and the advantages of selecting the largest divisor in elimination methods are demonstrated but the general round-off problem is not touched. High speed automatic digital computers are not discussed. In the final chapter there is a short introduction to abstract algebra: groups, rings, fields, etc.

All in all, the author has succeeded in his attempt to provide a transition between elementary algebra and the specialized fields of numerical analysis and modern algebra.

S. H. Crandall

Space-charge waves and slow electromagnetic waves. By A. H. W. Beck. Pergamon Press, New York, London, Paris, Los Angeles, 1958. xi + 396 pp. $15.00.

This book is the eighth volume in the International Series of Monographs on Electronics and Instrumentation. It covers a great deal of ground and on the whole the material is well presented, with more or less complete reference to pertinent published papers. There is a fair amount of mathematics used, although this entails not much more than a knowledge of the simpler properties of Bessel functions and some matrix algebra. There are some places where the logical structure is rather poor and this is complicated further by numerous misprints in some of the more mathematical sections. Also, the author occasionally quotes results from published papers in a highly condensed form without complete definitions of symbols used. This is again complicated by misprints and it was not always easy for the reviewer to follow the deductions without reference to the paper in question. However, these difficulties do not detract from the general worth of the book and the reviewer believes that the applied mathematician who wishes to become thoroughly acquainted with the theory underlying the various microwave valves in use today may do so by reading this book which is obviously written by one with an extensive knowledge of the subject. A condensed summary of the contents follows:

A short general introduction is given in Chapter I in which various types of amplifiers are briefly mentioned. Chapter II gives a condensed account of Maxwell's electromagnetic theory. Propagation modes are discussed for rectangular and circular waveguides and expressions for the power flows are obtained. Vector and scalar potentials and the Hertzian vectors are introduced. Chapter III deals with various slow wave structures and in particular with disk-loaded waveguides, interdigital delay lines and both sheath and tape helices. Their properties are discussed in relation to the Brillouin diagram (frequency versus phase constant). Chapter IV is devoted to space-charge wave theory and topics include the determination of the space-charge reduction factors for cylindrical and annular beams in cylindrical tunnels; space-charge waves on Brillouin beams and on confined beams; space-charge waves in accelerating fields and in crossed fields; the distribution function approach for multivelocity electron beams and plasma oscillations. Chapter V utilizes some of these results in discussing the matching of input conditions and space-charge waves on cylindrical, annular, Brillouin and confined beams, and on beams in helices.

Chapter VI is concerned with space-charge waves in klystrons while Chapter VII is devoted to travelling-wave tubes and backward-wave oscillators. Some crossed field devices are discussed in Chapter VIII while some special space-charge wave devices, such as the two-beam tube and the transverse current T.W.A. (travelling-wave amplifier), are mentioned in Chapter IX. The final Chapter deals with noise phenomena in space-charge wave devices and after discussing shot noise in diodes, the noise factor for valves with input resonators and the noise figure of T.W.A.'s, the general theory of noise in beams, based on the theorems of power flow in space-charge waves, is presented in the matrix formulation adopted by Haus and Robinson, although the proof of the noise invariants is omitted. The smoothing of current fluctuations near the potential minimum is also briefly discussed. There are twelve Appendices, mostly mathematical in content, and these are followed by a few questions on each chapter, a list of recent references, a list of the major symbols used throughout the book and, finally, the index.

J. A. MORRISON

Selected papers on quantum electrodynamics. Edited by Julian Schwinger. Dover Publications, Inc., New York, 1958. xvii + 424 pp. $2.45.

This book is a collection of thirty-four papers taken from the literature of quantum electrodynamics—papers which together with a preface summarize the present state of the theory. In summing up the problems still facing quantum electrodynamics the editor points out that "the real significance of the work of the past decade lies in the recognition of the ultimate problems facing quantum electrodynamics."

Papers are included in this collection by the following: Dirac, Fermi, Fock, Poldolsky, Jordan, Wigner, Heisenberg, Weisskopf, Block, Nordsieck, Foley, Kusch, Lamb, Retherford, Bethe, Schwinger, Oppenhemier, Tomonga, Pauli, Villars, Feynman, Dyson, Karplus, Klein, Källen, and Kroll.

Of these papers twenty-nine are in English, three in German, and one each in French and Italian.

R. TRUELL

Computability and unsolvability. By Martin Davis. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1958. xxv + 210 pp. $7.50.

This book is an introduction to the theory of computability and noncomputability, usually referred to as the theory of recursive functions, a branch of pure mathematics. In the light of recent developments in computers, decision problems, i.e., problems "which inquire as to the existence of an algorithm for deciding the truth or falsity of a class of statements," are potentially of interest to other than pure mathematicians. The work is an outgrowth of a graduate course at the University of Illinois and series of lectures given at the University of Illinois and the Bell Telephone Laboratories.

The major part of the book is self-contained and assumes no particular mathematical training on the part of the reader. A degree of mathematical maturity, in particular the ability to follow abstract proofs, is, however, necessary. Acquaintance with elementary mathematical logic is desirable.

The concept of Turing machine is made central to the development. Thus Turing's approach is combined with the methods of Gödel and Kleene to present the various aspects of the theory of computability in a unified manner. The general theory is applied to combinatorial problems, problems related to Hilbert's Tenth Problem, and systems of symbolic logic. The general theory is extended to include the Kleene hierarchy of arithmetical predicates, computable functionals, and the classifications of unsolvable decision problems.

A. A. GRAU

Modern geometrical optics. By Max Herzberger. Interscience Publishers, Inc., New York, London, 1958. xii + 504 pp. $15.00.

This work is both a treatise and a text on the methods of geometrical optics. As far as the reviewer is aware it is the only book of this kind in English (other than the notes of Lumeberg which were never published).

There are essentially two distinct sections of the book. The first part is concerned with the ray tracing and the calculation of optical systems; this consists of about the first thirteen chapters. Much of the second section of the book is concerned with the general laws of geometrical optics. The characteristic function methods of Hamilton, the general laws of image formation, the properties of concentric systems, and the properties of rotation symmetric systems are naturally an important part of the book.

The contents of the book are listed as follows:

Part I, Ray Tracing; Part II, Precalculation of Optical Systems; Part III, General Laws; Part IV, Concentric Systems; Part V, Rotation-Symmetric Systems; Part VI, Approximation Theory for Normal Systems; Part VII, Third and Fifth Order Image-Error Theory; Part VIII, Interpolation Theory of the Optical Image; Part IX, Optics in General Media; Part X, Appendix.

The only topics that the reviewer felt were noticeably missing, topics that might have been included, are those of integral invariants and perhaps something about electron optics; it may be that this is asking for too much.

The book represents a tremendous amount of work. It is well written, and it should be the standard work in this field for a long time.

R. TRUELL

Principles of quantum electrodynamics. Translated from the German by J. Bernstein, with additions and corrections by Walter E. Thirring. Academic Press Inc., New York, London, 1958. xv + 234 pp. $8.00.

This book is a text on the quantum electrodynamic part of field theory. It is more advanced in nature than the book of Wentzel entitled "Quantum Theory of Fields." The text is the result of an effort by the author to present mainly what cannot be found elsewhere in books on this subject. There are four parts of the text divided as follows:

I. General Introduction: Units and Orders of Magnitude; Classical Electrodynamics; General Formalism of Quantum Theory of Fields. II. Free Fields: General Discussion; Special Fields; Matrix Elements; Fluctuation Phenomena. III. Fields With External Sources: General Formulae; Emission of Light; The Dirac Field in an External Electric Field; The Limitations of Measurability. IV. Interacting Fields: General Orientation; Scattering Processes; Renormalization Theory; Higher Order Correction; Outlook.

Appendix I, Dirac Matrices. Appendix II, Green's Functions (Relativistic Wave Equation)

While this text is intended for the advanced student, there are parts such as section 16 of Part IV which will be of interest even to those who only want to know what the problems are at the present time.

R. TRUELL

Theorie schallnaher Strömungen, By K. G. Guderley. Springer-Verlag, Berlin, Gottingen, Heidelberg, 1957. xv + 376 pp. $10.07.

The study of transonic flow as a special field was stimulated by the failure of classical treatments of compressible flow, both by theoretical and experimental means, to describe flow phenomena in the Mach number range close to critical conditions. This gap arises because, near Mach number one, the linearized theory of compressible flow is invalid and normal High Speed Wind Tunnels become choked.

Much of the fundamental theory of transonic flow was developed by Dr. Guderley himself and it is appropriate that he should now present a connected account of his own and other contributions to the subject. Although his book is primarily concerned with transonic small disturbance theory based on Tricomi's equation it is self contained and the theory is carefully introduced with the necessary background of fundamental Gas Dynamics. The book will be of considerable interest to Applied Mathematicians and Theoretical Aerodynamicists. A great deal of space is given to special solutions of Tricomi's equation and their use in solving the problem of flow past a double wedge profile in a sonic stream. This is a field to which the author devoted many years of research, previously published in somewhat inaccessible form. It is certainly valuable to have this work in a single volume. The book would, in the reviewer's opinion, have been better balanced if more space had been given to other approaches to transonic small disturbance theory, notably by Oswatitsch, and to a fuller description of the purely numerical solution of transonic flow problems. The book is welcome as the first, and much needed, book on transonic flow and is strongly recommended to Fluid Dynamicists. There is still room, however, for a more complete account of the theory and also for an account of the difficult experimental work in this field. Further, since the book was written a number of transonic flow problems have been solved in the U. S. S. R. by Chushkin, using the successful numerical technique of Dorodnitsin for second order partial differential equations of mixed type. This approach certainly competes with the classical treatment based on Tricomi's equation. It is hoped that future editions of Dr. Guderley's book will include an account of this recent work.

There are 11 chapters in the book. Following a first chapter devoted to the basic principles of Steady Gas Dynamics the transonic similarity concept is introduced in Chapter II. The third chapter contains a brief account of Linearized Transonic Flow. In Chapter IV exact solutions of the simplified transonic potential equation are given for flow through a Laval nozzle and flow of a sonic jet. Chapter V gives a fairly complete account of the hodograph method leading to Tricomi's equation when transonic similarity is introduced. Chapter VI is largely a physical discussion of transonic flow phenomena in the hodograph plane. The author's special solutions of Tricomi's equation based on the singularity at the sonic point are described in full in Chapter VII. This is followed, naturally, by the solution of problems of flow with Mach number 1 in Chapter VIII. Chapter IX presents methods for determining flow fields with free stream Mach numbers slightly different from 1, based essentially on a linear perturbation of the solution given in Chapter VIII. In Chapter X some special points are discussed such as the reflection of disturbances at a sonic line. The final chapter is concerned with the extension of Guderley's basic theory to axially symmetrical flow.

M. Holt

Advances in applied mechanics. Editors: H. L. Dryden and Th. von Kármán. Academic Press, Inc., New York, 1958. x + 459 pp. $12.00.

The appearance of a new volume in this series, with its outstanding surveys of selected topics in Applied Mechanics, is always most welcome. For it is only by means of review articles of the type presented here that one can even attempt to keep abreast of developments in so inclusive a field as that of Applied Mechanics, an inclusiveness well demonstrated by the titles of the present articles: "Supersonic air ejectors" by J. Fabri and R. Siestrunck, "Unsteady airfoil theory" by A. I. Van De Vooren, "The theory of distributions" by Charles Saltzer, "Stress wave propagation in rods and beams" by H. N. Abramson, H. J. Plass and E. A. Ripperger, "Problems in hydromagnetics" by Edward A. Frieman and Russell M. Kulsrud, "Mechanics of granular media" by H. Deresiewicz, and "Condensation in in supersonic and hypersonic wind tunnels" by P. P. Wegener and L. M. Mack.

W. H. Reid

Notes on analog-digital conversion techniques. Edited by Alfred K. Susskind. The Technology Press of Mass. Institute of Technology, Cambridge, and John Wiley & Sons, Inc., New York, 1957. x + 877 pp. $10.00.

The articles, by various authors, in this book had their origin in a special summer program on the subject at M. I. T. in 1957. The subject matter is divided into three parts: systems analysis, engineering analysis of devices and a case study, of which the first bears most interest to mathematicians, in particular Chapter 2 on the theory of sampling and the theory of quantizing, and Chapter 3 on codes.

W. F. FREIBERGER

Elementary mathematical programming. By Robert W. Metzger. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1958. ix + 246 pp. $5.95.

This book is addressed to industrial engineers and others with little mathematical background who wish to know the mechanics of programming without being burdened with proofs or conceptual discussions. It is lucid and provides many practical illustrations but does not discuss the limitations of the very simplified models.

WALTER FREIBERGER

Notes on Operations Research 1959. Assembled by the Operations Research Center, M.I.T. The Technology Press, Massachusetts Institute of Technology, Cambridge, Mass., 1959. viii + 256 pp. $4.00.

These are the lecture notes for a special program in operations research that members of the Operations Research Center at M.I.T. conducted for persons from European NATO countries at the Center for Experimental Aerodynamics in Brussels in August 1959. The scope of the book is indicated by the following table of contents: Introduction (P. M. Morse)—Probability (G. P. Wadsworth)—Search (B. O. Koopman)—Markov Processes (P. M. Morse)—Queuing (H. P. Galliher)—Control Processes (R. A. Howard)—Organization of Operations Research Groups (P. M. Morse)—Sequential Decision Processes (G. E. Kimball and R. A. Howard)—Reliability and Maintenance (G. E. Kimball)—Information theory (B. O. Koopman and G. E. Kimball)—Production Scheduling (H. P. Galliher)—Simulation of Random Processes (H. P. Galliher)—Bibliography.

The measurement of power spectra. By R. B. Blackman and J. W. Tukey. Dover Publications, Inc., New York, 1959. x + 190 pp. $1.85.

This is a republication of "Power Spectra from the Point of View of Communications Engineering", which originally appeared in Volume 37 of the Bell System Technical Journal.

Les Principes de la Théorie Electromagnétique et de la Relativite. By Marie-Antoinette Tonnelat. Masson et Cie, Editeurs, Paris VI^{e}, 1959. 394 pp. $10.25.

This book is concerned with the study of the principles of classical and relativistic electromagnetic field theory and gravitational field theory. The discussion of the principles of electromagnetic theory includes among other things, discussion of the electron theory of Lorentz and the electrodynamics of bodies in motion. The fifth chapter begins the discussion of relativity and the remaining three quarters of the book does an excellent job of this. The only feature of the book that detracts from its value is the rather poor printing job, especially the tensor indices which are sometimes illegible.

ROHN TRUELL

Theory of functionals and of integral and integro-differential equations. By Vito Volterra. Dover Publications, New York, 1959. xiv + 226 pp. $1.75.

This is an unabridged republication of the English translation of Volterra's work published by Blackie & Son Ltd. in 1930. A preface by Professor G. C. Evans and a biography and bibliography by Sir Edmund Whittaker (originally published as an obituary note) have been added.

Plastic analysis of structures. By Philip G. Hodge, Jr. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1959. xiv + 364 pp. $10.50

The point of view that the rational analysis and design of structures composed of elastic-ductile materials requires inclusion of plastic action in the theory of structures has been attracting more interest recently, as evidenced by the increasing number of books published on the subject. The present book constitutes a systematic and up-to-date account of the theory of plastic analysis of structures and it is broader in coverage than its predecessors. Structures treated in the book include beams and frames in bending, beams under combined stresses, plates and shells and slabs with cut-outs. The subject matter is approached from the view-point of limit analysis. Emphasized are two basic theorems which enable lower and upper bounds on the load carrying capacity of structures to be determined. Part I, *Bending of Beams and Frames*, treats the application of plastic methods to frame-type structures. It also deals with such topics as elastic plastic deformations, variable and repeated loading and procedures of design for economy in materials. Part II, *Structures Under Combined Stresses*, is primarily concerned with combined stresses in beams, circular plates, and circular cylinderical shells. More general plate and shell problems and problems in plane stress have also been covered. Finally the book closes with a brief introduction to some of the problems encountered in the dynamic loading of plastic structures. References and Problems are given at the end of each chapter.

The clear presentation and reasonably elementary level of mathematical tools employed will undoubtedly make this text attractive to the interested students (both graduate and undergraduate) and also to the practicing engineer. However the structural engineer is probably too much elasticity-minded and may need additional convincing before accepting such concepts of plasticity theory as rigid-plastic action, plastic hinges and plastic collapse. The critical presentation and discussion of some experimental work, in addition to the test results given in the text, may prove helpful in this respect.

It may also be noted that questions concerning the effects of geometry changes and strain-hardening do not receive extensive attention in the book. The question of plastic stability is hardly considered in the book and beneficial effects of geometry changes and strain-hardening on the load carrying capacity of structures are mentioned only briefly. Some systematic discussion of these topics in the future editions of the book would increase the value of this interesting work.

E. T. ONAT

Notice of

SYMPOSIUM ON PLASTICITY

The Second Symposium on Naval Structural Mechanics will be held April 5, 6, and 7, 1960 at Brown University, Providence, R. I., under the joint sponsorship of the Office of Naval Research, Department of the Navy, and of Brown University. The Symposium will be devoted exclusively to the field of plasticity. The program will consist of critical surveys in selected areas and of reports on original research, with ample time for discussion. The organizing committee consists of Professors E. H. Lee and P. S. Symonds (co-chairmen), D. C. Drucker and W. Prager.