A DERIVATION OF THE BASIC EQUATIONS FOR HYDRODYNAMIC LUBRICATION WITH A FLUID HAVING CONSTANT PROPERTIES*

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Abstract. In this paper small parameter techniques are used to derive Reynolds' lubrication equations, and refinements thereof, from the full Navier-Stokes equation. An effort has been made to retain rigor in the development comparable to that used in present-day boundary-layer developments.

To derive the differential equations for flow in a curved film of arbitrary thickness requires the use of general tensor analysis. The mathematical manipulations are somewhat involved, but one of the results—a refined Reynolds equation—can be simply written for a journal or slipper bearing as follows:

$$\frac{\partial}{\partial x}\left\{h^3\left(1-\frac{h}{D}\right)\frac{\partial p}{\partial x}\right\}+\frac{\partial}{\partial z}\left\{h^3\left(1+\frac{h}{D}\right)\frac{\partial p}{\partial z}\right\}=6\mu U\frac{\partial}{\partial x}\left\{h\left(1-\frac{h}{3D}\right)\right\}.$$

Here:

D = shaft diameter (infinite for a slipper bearing)
h = film thickness
p = fluid pressure
U = shaft surface velocity
x = distance around shaft in direction of rotation
z = distance parallel to shaft axis
μ = fluid viscosity

The error of the above differential equation is of the order of $(h/L)^2$, where L is the film length in the direction x.

1. Introduction. The purpose of this paper is to provide a derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties. It is expected that analytical techniques similar to those employed here can be adapted to the development of equations applicable to fluids having pressure- and temperature-dependent properties.

The derivation given here applies directly to the geometries of both journal and slipper bearings. No difficulties are anticipated in the application of similar analysis to other geometries, such as those which can occur in thrust bearings, but the writer wished to avoid becoming lost in too much generality.

In the literature, a close approach to the present work appears in an article by Wannier [1]. The present work extends that of Wannier in three respects. First, the complete Navier-Stokes equations are used as a starting-point (rather than "Stokes" equations). Second, the mean film surface can have finite curvature. Third, the present approximation procedure is susceptible to improvement without regression.

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*Received June 4, 1958.

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Figure 1 shows the physical situation to be analyzed. A shaft rotates within a bearing, which, for the purposes of analysis, is only partial. A circular shaft of radius, R, rotates at angular velocity, ω. The small film thickness, h, between this shaft and the bearing surface can be an arbitrary function of position. It is completely filled with a fluid possessing constant properties, free to flow in and out of the film where it is exposed to ambient conditions at the edge of the bearing.

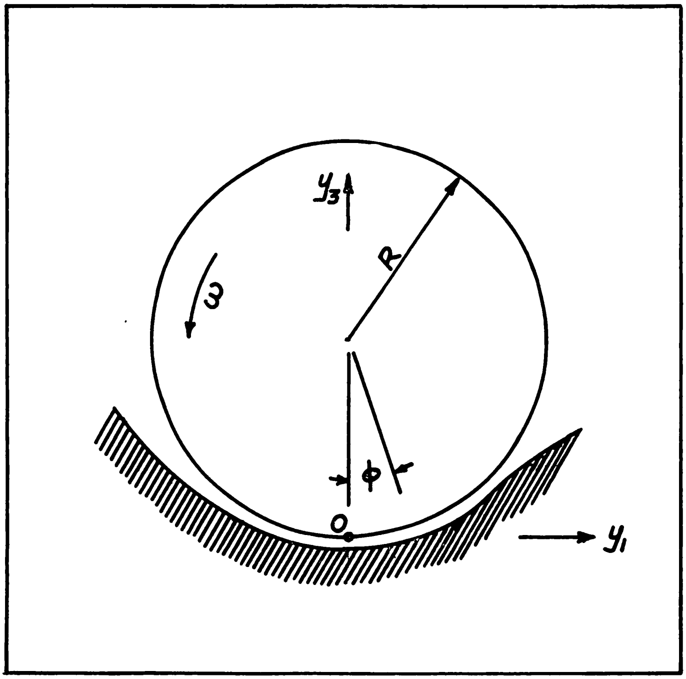

Fig. 1—Schematic diagram of bearing

The problem to be posed here is that of predicting the velocity and pressure distributions within the film. This problem can perhaps best be stated in terms of the variables:

$$\xi^1 \equiv \frac{R}{L}\varphi; \qquad \xi^2 \equiv \frac{y^2}{L}; \qquad \xi^3 \equiv \frac{R - r}{h(\xi^1, \xi^2)}, \tag{1.1}$$

where L is a characteristic dimension of the bearing. The position of a fluid particle is known when its coordinates ξ^1, ξ^2 and ξ^3 are specified. Likewise, its motion is given by the time derivatives of these quantities, u^i. Thus:

$$u^i \equiv \frac{d\xi^i}{dt}. \tag{1.2}$$

Then it is convenient to define the following set of dimensionless "velocities"

$$u_*^i \equiv \frac{L^2 u^i}{\nu}. \tag{1.3}$$

Corresponding to Fig. 1 we note that:

$$u_*^i = 0; \quad \xi^3 = -1, \quad i = 1, 2, 3.$$

$$u_*^1 = \frac{LR\omega}{\nu}; \quad u_*^i = 0, \quad i = 2, 3; \quad \xi^3 = 0.$$

We can presume that the local film thickness, h, is given by:

$$h = h_0 \exp \{\varphi(\xi^1, \xi^2)\}, \tag{1.4}$$

where the function φ and its derivatives are $0(1)$. The value of h_0 sets the magnitude, so to speak, of the film thickness throughout the bearing

Finally, we introduce a dimensionless "pressure" π such that

$$\pi \equiv \frac{p}{\rho\left(\frac{\nu}{h_0}\right)^2}. \tag{1.5}$$

The value of π is specified around the periphery of the film, and may there generally be taken as zero.

The significant feature of lubrication hydrodynamics is the smallness of the dimensionless parameter

$$\epsilon \equiv h_0/L. \tag{1.6}$$

Now it is possible to imagine at least two experimental sequences in which this parameter becomes progressively smaller, while at the same time the boundary conditions on the u_*^i and π remain invariant. In the first of these sequences, L is progressively increased with h_0 constant, while R is varied proportionately to L and ω is varied as L^{-2}. Thus u_*^1 remains constant on the shaft, and π on the bearing periphery is unaltered if the external pressure distribution is unaltered. In the second sequence, h_0 is progressively reduced and fluids of equal kinematic viscosity "ν" are employed which have, nevertheless, progressively lower densities; i.e., $\rho \sim h_0^2$. (In the event that the peripheral pressure is constant, as is usual, then this constant pressure can be taken as datum, and the fluid density need not be changed in this second sequence.) Again, the boundary conditions on the u_*^i and π are invariant. The existence of such hypothetical experimental sequences is not necessary to the use of (h_0/L) as a small parameter in a mathematical expansion, but it does tend to assure that there will be a range of experimental conditions for which the early terms of the expansion will provide an adequate description of the observed variables.

Although the boundary conditions of the problem can be made independent of ϵ, as shown above, the differential equations for the u_*^i and π in terms of ξ^1, ξ^2, ξ^3, do contain ϵ in such a manner that a non-singular perturbation problem can be formulated. Thus, we hypothesize that we can represent the dependent variables as follows:

$$u_*^i = U_0^i(\xi^1, \xi^2, \xi^3) + \epsilon U_1^i(\xi^1, \xi^2, \xi^3) + \epsilon^2 U_2^i(\xi^1, \xi^2, \xi^3) \quad \text{etc.}$$
$$\pi = P_0(\xi^1, \xi^2, \xi^3) + \epsilon P_1(\xi^1, \xi^2, \xi^3) + \epsilon^2 P_2(\xi^1, \xi^2, \xi^3). \tag{1.7}$$

When these series are substituted into the complete Navier-Stokes equations, and the coefficients of the various powers of ϵ are equated to zero, a sequence of differential equations is generated for the U_k^i and P_k . The lowest order equations are the original equations of Reynolds. Successively better solutions can be found by solving the higher-order differential equations in a manner explicitly indicated in this paper. If the series in ϵ [Eq. (1.7)] converge, true solutions of the Navier-Stokes equation should result.

2. The metric tensor. The simplest manner of obtaining the differential equations for the u_*^i and π is through the use of tensor calculus. Hence we first compute the covariant components of the metric tensor. They are given by:

$$g_{\alpha\beta} = \sum_{i=1}^{i=3} \frac{\partial y^i}{\partial \xi^\alpha} \frac{\partial y^i}{\partial \xi^\beta}. \tag{2.1}$$

The intermediate algebraic details required to obtain the $g_{\alpha\beta}$ are recorded in the Appendix. The resulting matrix is given below

$$\frac{g_{\alpha\beta}}{L^2} = G_{\alpha\beta} = \begin{vmatrix} \left(1 - \xi^3 \frac{h}{R}\right)^2 + \left(\frac{\xi^3}{L} \frac{\partial h}{\partial \xi^1}\right)^2 & \left(\frac{\xi^3}{L}\right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^1} \\ \left(\frac{\xi^3}{L}\right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} & 1 + \left(\frac{\xi^3}{L} \frac{\partial h}{\partial \xi^2}\right)^2 & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^2} \\ \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^1} & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^2} & \left(\frac{h}{L}\right)^2 \end{vmatrix}. \tag{2.2}$$

The determinant of the matrix $| g_{\alpha\beta} |$ is:

$$g \equiv | g_{\alpha\beta} | = \left(1 - \xi^3 \frac{h}{R}\right)^2 L^4 h^2. \tag{2.3}$$

The contravariant components of the metric tensor are found from the formula

$$g^{\alpha\beta} = \frac{1}{g} \cdot \text{Cofactor of } g_{\beta\alpha} \text{ in } g. \tag{2.4}$$

As shown in the Appendix, the contravariant array is as follows

$$L^2 g^{\alpha\beta} = G^{\alpha\beta} = \begin{vmatrix} \dfrac{1}{\left(1 - \xi^3 \frac{h}{R}\right)^2} & 0 & \dfrac{\xi^3 \frac{\partial \ln h}{\partial \xi^1}}{\left(1 - \xi^3 \frac{h}{R}\right)^2} \\ 0 & 1 & -\xi^3 \dfrac{\partial \ln h}{\partial \xi^2} \\ \dfrac{-\xi^3 \frac{\partial \ln h}{\partial \xi^1}}{\left(1 - \xi^3 \frac{h}{R}\right)^2} & -\xi^3 \dfrac{\partial \ln h}{\partial \xi^2} & \dfrac{L^2}{h^2} + \dfrac{\left(\xi^3 \frac{\partial \ln h}{\partial \xi^1}\right)^2}{\left(1 - \xi^3 \frac{h}{R}\right)^2} + \left(\xi^3 \dfrac{\partial \ln h}{\partial \xi^2}\right)^2 \end{vmatrix}. \tag{2.5}$$

Finally, the Euclidean Christoffel symbols are required. They are given by the formula:

$$\Gamma^i_{\alpha\beta}(\xi^1, \xi^2, \xi^3) = \tfrac{1}{2} g^{i\sigma} \left(\frac{\partial g_{\sigma\beta}}{\partial \xi^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial \xi^\beta} - \frac{\partial g_{\alpha\beta}}{\partial \xi^\sigma}\right) \tag{2.6}$$

or:

$$\Gamma^i_{\alpha\beta}(\xi^1, \xi^2, \xi^3) = \tfrac{1}{2} G^{i\sigma} \left(\frac{\partial G_{\sigma\beta}}{\partial \xi^\alpha} + \frac{\partial G_{\alpha\sigma}}{\partial \xi^\beta} - \frac{\partial G_{\alpha\beta}}{\partial \xi^\sigma}\right). \tag{2.7}$$

Explicit expressions for the $\Gamma^i_{\alpha\beta}$ will not be given; rather we shall determine their orders-of-magnitude in terms of ϵ.

We first note that all $G^{\alpha\beta}$ are $0(\epsilon^0)$, with the exception of G^{33}, which is $0(\epsilon^{-2})$. Thus:

$$G^{\alpha\beta} = 0\{\exp(-2\delta_3^\alpha\ \delta_3^\beta \ln \epsilon)\}. \tag{2.8}$$

Next we note that all derivatives of the $G_{\alpha\beta}$ are $0(\epsilon^2)$, with the exception of the derivatives of G_{11}, which are $0(\epsilon)$. Thus:

$$\frac{\partial G_{\alpha\beta}}{\partial \xi^k} = 0\{\exp(2 - \delta_\alpha^1\ \delta_\beta^1) \ln \epsilon)\}. \tag{2.9}$$

Substituting Eqs. (2.8) and (2.9) into Eq. (2.7), we obtain:

$$\begin{aligned}\Gamma^i_{\alpha\beta} = 0\{&\exp[(-2\ \delta_3^i\ \delta_3^\sigma + 2 - \delta_\sigma^1\ \delta_\beta^1) \ln \epsilon]\}\\ &+ 0\{\exp[(-2\ \delta_3^i\ \delta_3^\sigma + 2 - \delta_\alpha^1\ \delta_\sigma^1) \ln \epsilon]\}\\ &+ 0\{\exp[(-2\ \delta_3^i\ \delta_3^\sigma + 2 - \delta_\alpha^1\ \delta_\beta^1) \ln \epsilon]\} \qquad \sigma = 1, 2, 3\end{aligned} \tag{2.10}$$

By picking the lowest power of ϵ for each $\Gamma^i_{\alpha\beta}$, we reach the following conclusions.

$$\left.\begin{array}{l} i \neq 3,\quad \alpha \neq 1,\quad \beta \neq 1: \qquad \Gamma^i_{\alpha\beta} = 0(\epsilon^2)\\ i \neq 3,\quad \alpha = 1,\quad \text{or}\quad \beta = 1,\quad \text{or}\quad \alpha = \beta = 1: \qquad \Gamma^i_{\alpha\beta} = 0(\epsilon)\\ i = 3,\quad \alpha \neq 1,\quad \text{or}\quad \beta \neq 1: \qquad \Gamma^i_{\alpha\beta} = 0(\epsilon^0)\\ i = 3,\quad \alpha = \beta = 1: \qquad \Gamma^i_{\alpha\beta} = 0(\epsilon^{-1}). \end{array}\right\}. \tag{2.11}$$

3. Reduction of the momentum and mass continuity equations. The Navier-Stokes equations for a fluid having constant properties are [2]:

$$\begin{aligned}0 = \nu g^{\alpha\beta}\Bigg[&\frac{\partial^2 u^i}{\partial\xi^\alpha\, \partial\xi^\beta} + \Gamma^i_{\sigma\beta}\frac{\partial u^\sigma}{\partial \xi^\alpha} + \Gamma^i_{\sigma\alpha}\frac{\partial u^\sigma}{\partial\xi^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial u^i}{\partial\xi^\sigma} + \left(\frac{\partial \Gamma^i_{\sigma\alpha}}{\partial\xi^\beta} + \Gamma^i_{\tau\beta}\Gamma^\tau_{\sigma\alpha} - \Gamma^i_{\sigma\tau}\Gamma^\tau_{\alpha\beta}\right)u^\sigma\Bigg]\\ &- u^\alpha\left(\frac{\partial u^i}{\partial \xi^\alpha} + \Gamma^i_{\sigma\alpha}u^\sigma\right) - \frac{1}{\rho}\, g^{i\alpha}\,\frac{\partial p}{\partial \xi^\alpha}.\end{aligned} \tag{3.1}$$

In terms of the dimensionless velocities u_*^i and the dimensionless pressure, π, this equation becomes:

$$\begin{aligned}0 = G^{\alpha\beta}\Bigg[&\frac{\partial^2 u^i}{\partial\xi^\alpha\, \partial\xi^\beta} + \Gamma^i_{\sigma\beta}\frac{\partial u_*^\sigma}{\partial \xi^\alpha} + \Gamma^i_{\sigma\alpha}\frac{\partial u_*^\sigma}{\partial\xi^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial u_*^i}{\partial\xi^\sigma}\\ &+ \left(\frac{\partial \Gamma^i_{\sigma\alpha}}{\partial\xi^\beta} + \Gamma^i_{\tau\beta}\Gamma^\tau_{\sigma\alpha} - \Gamma^i_{\sigma\tau}\Gamma^\tau_{\alpha\beta}\right)u_*^\sigma\Bigg]\\ &- u_*^\alpha\left(\frac{\partial u_*^i}{\partial \xi^\alpha} + \Gamma^i_{\sigma\alpha}u_*^\sigma\right) - \epsilon^{-2} G^{i\alpha}\,\frac{\partial \pi}{\partial \xi^\alpha}.\end{aligned} \tag{3.2}$$

With $i \neq 3$, we retain in Eq. (3.2) those terms of lowest power in ϵ; i.e., of $0(\epsilon^{-2})$ and $0(\epsilon^{-1})$. The resulting, simplified equation is:

$$\begin{aligned}0 = -G^{11}\Gamma^3_{11}\frac{\partial u_*^i}{\partial\xi^3} + G^{33}\left[\frac{\partial^2 u_*^i}{(\partial\xi^3)^2} + 2\Gamma^i_{13}\frac{\partial u_*^1}{\partial\xi^3} + \Gamma^3_{33}\frac{\partial u_*^i}{\partial\xi^3} + \frac{\partial\Gamma^i_{13}}{\partial\xi^3}u_*^1\right]\\ -\epsilon^{-2}G^{i\alpha}\frac{\partial\pi}{\partial\xi^\alpha}.\end{aligned} \tag{3.3}$$

Explicit expressions for the $G^{\alpha\beta}$ are given by (2.5). The required $\Gamma^{\alpha}_{\beta\gamma}$ in Eq. (3.3) are given with sufficient accuracy as:

$$\Gamma^{i}_{13} = \tfrac{1}{2}G^{i1}\frac{\partial G_{11}}{\partial \xi^3} = \tfrac{1}{2}G^{i1}\left(-2\frac{h}{R}\right) = -G^{i1}\frac{h}{R} = 0(\epsilon), \tag{3.4}$$

$$\frac{\partial \Gamma^{i}_{13}}{\partial \xi^3} = 0(\epsilon^2) \quad \text{because} \quad \frac{\partial h}{\partial \xi^3} = 0. \tag{3.5}$$

$$\Gamma^{3}_{33} = \tfrac{1}{2}G^{3\sigma}\left[2\frac{\partial G_{\sigma 3}}{\partial \xi^3} - \frac{\partial G_{33}}{\partial \xi^\sigma}\right] = 0(\epsilon^2) \quad \text{because} \quad \frac{\partial G_{33}}{\partial \xi^3} = 0 \tag{3.6}$$

$$\Gamma^{3}_{11} = -\tfrac{1}{2}G^{33}\frac{\partial G_{11}}{\partial \xi^3} = \frac{h}{R}\left(\frac{L}{h}\right)^2 = \frac{L}{R}\left(\frac{L}{h}\right) = 0\left(\frac{1}{\epsilon}\right). \tag{3.7}$$

By virtue of these relations, Eq. (3.3) becomes:

$$0 = \left(\frac{L}{h}\right)^2\left[\frac{\partial^2 u^i_*}{(\partial \xi^3)^2} - 2G^{i1}\frac{h}{R}\frac{\partial u^1_*}{\partial \xi^3} - \frac{h}{R}\frac{\partial u^i_*}{\partial \xi^3}\right] - \epsilon^{-2}\left[G^{i1}\frac{\partial \pi}{\partial \xi^1} + G^{i2}\frac{\partial \pi}{\partial \xi^2} + G^{i3}\frac{\partial \pi}{\partial \xi^3}\right]. \tag{3.8}$$

With $i = 3$, we again seek the two lowest powers of ϵ in Eq. (3.2). It is readily found that, correct in terms of $0(\epsilon^{-4})$ and $0(\epsilon^{-3})$,

$$\frac{\partial \pi}{\partial \xi^3} = 0. \tag{3.9}$$

Now we examine the mass-continuity equation. In tensor form, it can be written as:

$$\frac{\partial}{\partial \xi^\alpha}(g^{\frac{1}{2}}u^\alpha) = 0, \tag{3.10}$$

where, of course,

$$g \equiv \left(1 - \xi^3\frac{h}{R}\right)^2 L^4 h^2. \tag{3.11}$$

Since u^3_* vanishes at $\xi_3 = 0$ and $\xi_3 = -1$, we can integrate Eq. (3.10) to get:

$$\frac{\partial}{\partial \xi^1}\int_0^{-1}\left(1 - \xi^3\frac{h}{R}\right)hu^1_*\,d\xi^3 + \frac{\partial}{\partial \xi^2}\int_0^{-1}\left(1 - \xi^3\frac{h}{R}\right)hu^2_*\,d\xi^3 = 0. \tag{3.12}$$

This last equation is exact.

4. Reynolds equation with first correction terms. In this section we shall obtain a Reynolds equation valid to $0(h_0/L)$. From Eqs. (3.8) we obtain:

$$\frac{\partial^2 u^1_*}{\partial(\xi^3)^2} - 2G^{11}\frac{h}{R}\frac{\partial u^1_*}{\partial \xi^3} - \frac{h}{R}\frac{\partial u^1_*}{\partial \xi^3} = G^{11}\left(\frac{h}{h_0}\right)^2\frac{\partial \pi}{\partial \xi^1}, \tag{4.1}$$

$$\frac{\partial^2 u^2_*}{\partial(\xi^3)^2} - \frac{h}{R}\frac{\partial u^2_*}{\partial \xi^3} = G^{22}\left(\frac{h}{h_0}\right)^2\frac{\partial \pi}{\partial \xi^2}, \tag{4.2}$$

where:

$$G^{11} = \frac{1}{\left(1 - \xi^3\dfrac{h}{R}\right)^2} = 1 + 2\xi^3\frac{h}{R} + \cdots \tag{4.3}$$

$$G^{22} = 1. \tag{4.4}$$

Retention of terms through $0(\epsilon)$ in Eqs. (4.1) and (4.2) gives:

$$\frac{\partial^2 u_*^1}{\partial(\xi^3)^2} - 3\frac{h}{R}\frac{\partial u_*^1}{\partial \xi^3} = \left(1 + 2\xi^3\frac{h}{R}\right)\left(\frac{h}{h_0}\right)^2\frac{\partial \pi}{\partial \xi^1} \tag{4.5}$$

and

$$\frac{\partial^2 u_*^2}{\partial(\xi^3)^2} - \frac{h}{R}\frac{\partial u_*^2}{\partial \xi^3} = \left(\frac{h}{h_0}\right)^2\frac{\partial \pi}{\partial \xi^2}. \tag{4.6}$$

Because, to the degree of approximation presently being retained, π is independent of ξ^3, Eqs. (4.5) and (4.6) can be integrated with respect to ξ^3. For u_*^1 we obtain:

$$u_*^1 = \left(\frac{LR\omega}{\nu} + \frac{q_1\xi^3}{2}\right)(1 + \xi^3) + \frac{3h}{R}\left[\xi^3\left\{\frac{LR\omega}{2\nu} - \frac{q_1}{36}\right\} + (\xi^3)^2\left\{\frac{LR\omega}{2\nu} + \frac{q_1}{4}\right\} + (\xi^3)^3\frac{5q_1}{18}\right], \tag{4.7}$$

where

$$q_1 \equiv \left(\frac{h}{h_0}\right)^2\frac{\partial \pi}{\partial \xi^1}$$

and terms of $0(\epsilon^2)$ and higher have been neglected.

For u_*^2 we obtain:

$$u_*^2 = q_2\left[\frac{\xi^3(1 + \xi^3)}{2} + \frac{h}{R}\left\{\frac{\xi^3}{12} + \frac{(\xi^3)^2}{4} + \frac{(\xi^3)^3}{6}\right\}\right], \tag{4.8}$$

where

$$q_2 \equiv \left(\frac{h}{h_0}\right)^2\frac{\partial \pi}{\partial \xi^2}.$$

The "velocities" u_*^1 and u_*^2 from the above Eqs. (4.7) and (4.8) can now be substituted into Eq. (3.12) and the definite integrations called for in this last equation can be carried out explicitly. When terms through $0(\epsilon)$ are retained, the result is:

$$\frac{\partial}{\partial \xi^1}\left(\frac{h}{h_0}\right)\left[\left(\frac{h}{h_0}\right)^2\left(1 - \frac{h}{2R}\right)\frac{\partial \pi}{\partial \xi^1} - \left(6 - \frac{h}{R}\right)\frac{LR\omega}{\nu}\right] + \frac{\partial}{\partial \xi^2}\left(\frac{h}{h_0}\right)\left[\left(\frac{h}{h_0}\right)^2\left(1 + \frac{h}{2R}\right)\frac{\partial \pi}{\partial \xi^2}\right] = 0. \tag{4.9}$$

In more conventional variables, this equation becomes:

$$\frac{\partial}{\partial x}\left\{h^3\left(1 - \frac{h}{D}\right)\frac{\partial p}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{h^3\left(1 + \frac{h}{D}\right)\frac{\partial p}{\partial z}\right\} = 6\mu U\frac{\partial}{\partial x}\left\{h\left(1 - \frac{h}{3D}\right)\right\}, \tag{4.10}$$

where D is the shaft diameter, x is distance around the shaft, and z is distance along its axis.

In all usual applications, $h/D \ll 1$, so that Eq. (4.10) is essentially Reynolds' Lubrication Equation with correction terms. It has been derived in this paper from the full Navier-Stokes equation by a systematic procedure employing $\epsilon \equiv h_0/L$ as a small parameter. Now:

$$\frac{h}{D} = \frac{h}{h_0}\cdot\frac{h_0}{L}\cdot\frac{L}{D} = \epsilon\left(\frac{h}{h_0}\right)\left(\frac{L}{D}\right). \tag{4.11}$$

When the characteristic length L is maintained constant and D is taken progressively larger, $h/D \to 0$, and the equation for a plane slipper bearing is obtained. It is seen that Reynolds' equation for a plane slipper bearing has an error of $0(\epsilon^2)$, rather than $0(\epsilon)$.

5. Development of solutions to the Navier-Stokes equations. The general approximation procedure, of which we have just examined the first few steps, will now be outlined.

Substitution of the series 1.7 into Eq. (3.8) gives the following sets of equations which result from equating to zero the coefficients of the successive powers of "ϵ"

$$\frac{\partial^2 U_k^i}{\partial(\xi^3)^2} = \left(\frac{h}{h_0}\right)^2 \frac{\partial P_k}{\partial \xi^1} + H_{k-1}(\xi^1, \xi^2, \xi^3), \tag{5.1}$$

$$\frac{\partial^2 U_k^2}{\partial(\xi^3)^2} = \left(\frac{h}{h_0}\right)^2 \frac{\partial P_k}{\partial \xi^2} + I_{k-1}(\xi^1, \xi^2, \xi^3), \tag{5.2}$$

$$\frac{\partial P_k}{\partial \xi^3} = K_{k-1}(\xi^1, \xi^2, \xi^3). \tag{5.3}$$

In these equations, the functions H_{k-1}, I_{k-1} and K_{k-1} are known functions of space involving velocity and pressure functions of lower order obtained earlier in the approximation procedure. For example, H_{k-1} (ξ^1, ξ^2, ξ^3) would involve

$$U_{k-1}^1, U_{k-1}^2, U_{k-1}^3; \qquad U_{k-2}^1, U_{k-2}^2, \text{ etc.}$$

$$P_{k-1}, P_{k-2}, \text{ etc.}$$

and their derivatives.

Now from Eq. (5.3) we have:

$$P_k(\xi^1, \xi^2, \xi^3) = P_k(\xi^1, \xi^2, 0) + \int_0^{\xi^3} \frac{\partial P_k}{\partial \xi^3}\, d\xi^3. \tag{5.4}$$

Or:

$$\frac{\partial P_k}{\partial \xi^1} = \frac{\partial P_k(0)}{\partial \xi^1} + L_{k-1}(\xi^1, \xi^2, \xi^3). \tag{5.5}$$

In this equation, and hereafter, we shall abbreviate

$$P_k(0) \equiv P_k(\xi^1, \xi^2, 0).$$

Substitution of Eq. (5.5) into Eq. (5.1) gives:

$$\frac{\partial^2 U_k^1}{\partial(\xi^3)^2} = \left(\frac{h}{h_0}\right)^2 \frac{\partial P_k(0)}{\partial \xi^1} + M_{k-1}(\xi^1, \xi^2, \xi^3). \tag{5.6}$$

This last equation can be integrated twice with respect to ξ^3. For $k > 0$, the boundary conditions are:

$$U_k^i(\xi^1, \xi^2, 0) = U_k^1(\xi^1, \xi^2, -1) = 0. \tag{5.7}$$

After integration we have:

$$U_k^1(\xi^1, \xi^2, \xi^3) = \left(\frac{h}{h_0}\right)^2 \frac{\partial P_k(0)}{\partial \xi^1} \frac{(\xi^3)^2}{2} + N_{k-1}(\xi^1, \xi^2, \xi^3). \tag{5.8}$$

A like equation is obtained for U_k^2. Thus:

$$U_k^2(\xi^1, \xi^2, \xi^3) = \left(\frac{h}{h_0}\right)^2 \frac{\partial P_k(0)}{\partial \xi^2} \frac{(\xi^3)^2}{2} + O_{k-1}(\xi^1, \xi^2, \xi^3). \tag{5.9}$$

The mass-continuity equation, Eq. (3.12), gives the following recursion equation involving the U_k^i.

$$\frac{\partial}{\partial \xi^1} \int_0^{-1} hU_k^1(\xi^1, \xi^2, \xi^3)\, d\xi^3 + \frac{\partial}{\partial \xi^2} \int_0^{-1} hU_k^2(\xi^1, \xi^2, \xi^3) = Q_{k-1}(\xi^1, \xi^2). \tag{5.10}$$

Substitution of the U_k^1 and U_k^2 from Eqs. (5.8) and (5.9) gives:

$$\frac{\partial}{\partial \xi^1}\left(\frac{h}{h_0}\right)^3 \frac{\partial P_k(0)}{\partial \xi^1} + \frac{\partial}{\partial \xi^2}\left(\frac{h}{h_0}\right)^3 \frac{\partial P_k(0)}{\partial \xi^2} = R_{k-1}(\xi^1, \xi^2). \tag{5.11}$$

Here we have a partial differential equation for $P_k(0)$ as a function of ξ^1 and ξ^2. It is the "diffusion equation" with a variable "diffusion coefficient" $(h/h_0)^3$ and a source term, R_{k-1}. For $k > 0$ the boundary condition to be imposed at the edge of the fluid film is $P_k(0) = 0$.

In principle, Eq. (5.11) can be solved. Then P_k (ξ^1, ξ^2, ξ^3) can be found from Eq. (5.4) and U_k^1 (ξ^1, ξ^2, ξ^3) and U_k^2 (ξ^1, ξ^2, ξ^3) from Eqs. (5.8) and (5.9). All information for repeating the same process for U_{k+1}^1, U_{k+1}^2 and P_{k+1} is now available. Provided that the series (1.7) converge, solutions are developed for the complete Navier-Stokes equations subject to the boundary conditions

$$u_*^1 = U_0^1 = \frac{LR\omega}{\nu}; \qquad \xi^3 = 0$$

$$\pi = P_0 = f(t),$$

where the edge of the fluid film is described by the parametric equations

$$\xi^1 = \xi^1(t); \qquad \xi^2 = \xi^2(t).$$

Acknowledgment. This work was performed at the Franklin Institute, Philadelphia, Penna., under contract Nonr-2342 (00) Task NR 097–343. Jointly supported by the Department of Defense, Atomic Energy Commission, Maritime Administration. Administered by Office of Naval Research, Department of the Navy.

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Appendix. *Calculation of covariant components of metric tensor.* The transformation inverse to Eqs. (1.1) is:

$$y^1 = r \sin \varphi = (R - \xi^3 h) \sin \varphi = (R - \xi^3 h) \sin\left(\frac{L\xi^1}{R}\right), \tag{A.1}$$

$$y^2 = L\xi^2, \tag{A.2}$$

$$y^3 = R - r \cos \varphi = R - (R - \xi^3 h) \cos\left(\frac{L\xi^1}{R}\right). \tag{A.3}$$

From Eqs. (A.1–A.3):

$$\frac{\partial y^1}{\partial \xi^1} = \frac{L}{R}(R - \xi^3 h)\cos\varphi - \xi^3 \frac{\partial h}{\partial \xi^1}\sin\varphi, \tag{A.4}$$

$$\frac{\partial y^1}{\partial \xi^3} = -\xi^3 \frac{\partial h}{\partial \xi^2}\sin\varphi, \tag{A.5}$$

$$\frac{\partial y^1}{\partial \xi^3} = -h\sin\varphi, \tag{A.6}$$

$$\frac{\partial y^2}{\partial \xi^1} = 0; \qquad \frac{\partial y^2}{\partial \xi^2} = L; \qquad \frac{\partial y^2}{\partial \xi^3} = 0, \tag{A.7}$$

$$\frac{\partial y^3}{\partial \xi^1} = \frac{L}{R}(R - \xi^3 h)\sin\varphi + \xi^3 \frac{\partial h}{\partial \xi^1}\cos\varphi, \tag{A.8}$$

$$\frac{\partial y^3}{\partial \xi^2} = \xi^3 \frac{\partial h}{\partial \xi^2}\cos\varphi, \tag{A.9}$$

$$\frac{\partial y^3}{\partial \xi^3} = h\cos\varphi. \tag{A.10}$$

From the foregoing derivatives the covariant components $g_{\alpha\beta}$ can be calculated according to Eq. (2.1). For example:

$$\begin{aligned} g_{11} = {} & \left\{\frac{L}{R}(R - \xi^3 h)\right\}^2 \cos^2\varphi + \left(\xi^3 \frac{\partial h}{\partial \xi^1}\right)^2 \sin^2\varphi - 2\frac{L}{R}(R - \xi^3 h)\xi^3 \frac{\partial h}{\partial \xi^1}\sin\varphi\cos\varphi \\ & + \left\{\frac{L}{R}(R - \xi^3 h)\right\}^2 \sin^2\varphi + \left(\xi^3 \frac{\partial h}{\partial \xi^1}\right)^2 \cos^2\varphi + 2\frac{L}{R}(R - \xi^3 h)\xi^3 \frac{\partial h}{\partial \xi^1}\sin\varphi\cos\varphi \end{aligned} \tag{A.11}$$

or:

$$g_{11} = \left\{\frac{L}{R}(R - \xi^3 h)\right\}^2 + \left(\xi^3 \frac{\partial h}{\partial \xi^1}\right)^2. \tag{A.12}$$

Next:

$$g_{22} = \left(\xi^3 \frac{\partial h}{\partial \xi^2}\right)^2 \sin^2\varphi + L^2 + \left(\xi^3 \frac{\partial h}{\partial \xi^2}\right)^2 \cos^2\varphi = L^2 + \left(\xi^3 \frac{\partial h}{\partial \xi^2}\right)^2. \tag{A.13}$$

The other $g_{\alpha\beta}$ can be found by similar summations. All are combined in the matrix appearing as Eq. (2.2). The third-order determinant of this matrix is readily found to be

$$| g_{\alpha\beta} | = \left(1 - \xi^3 \frac{h}{R}\right) L^4 h^2. \tag{A.14}$$

Calculation of contravariant components of metric tensor. Equation (2.4) is used to calculate the contravariant components, $g^{\alpha\beta}$. For example, g^{33} is found as follows:

$$g^{33} = (-1)^{3+3} L^4 \left[\left(1 - \xi^3 \frac{h}{R}\right)^2 L^4 h^2\right]^{-1} \begin{vmatrix} \left(1 - \xi^3 \frac{h}{R}\right)^2 + \left(\frac{\xi^3}{L}\frac{\partial h}{\partial \xi^1}\right)^2 & \left(\frac{\xi^3}{L}\right)^2 \frac{\partial h}{\partial \xi^1}\frac{\partial h}{\partial \xi^2} \\ \left(\frac{\xi^3}{L}\right)^2 \frac{\partial h}{\partial \xi^1}\frac{\partial h}{\partial \xi^2} & 1 + \left(\frac{\xi^3}{L}\frac{\partial h}{\partial \xi^2}\right)^2 \end{vmatrix} \tag{A.15}$$

or:

$$g^{33} = \frac{1}{h^2} + \frac{1}{L^2\left(1 - \xi^3 \frac{h}{R}\right)^2} \left(\xi^3 \frac{\partial lnh}{\partial \xi^1}\right)^2 + \frac{1}{L^2} \left(\xi^3 \frac{\partial lnh}{\partial \xi^2}\right)^2. \tag{A.16}$$

Likewise, we find that:

$$g^{23} = (-1)^{2+3} L^4 \left[\left(1 - \xi^3 \frac{h}{R}\right)^2 L^4 h^2\right]^{-1} \begin{vmatrix} \left(1 - \xi^3 \frac{h}{R}\right)^2 + \left(\frac{\xi^3}{L} \frac{\partial h}{\partial \xi^1}\right)^2 & \left(\frac{\xi^3}{L}\right)^2 \frac{\partial h}{\partial \xi^1} \frac{\partial h}{\partial \xi^2} \\ \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^1} & \xi^3 \frac{h}{L^2} \frac{\partial h}{\partial \xi^2} \end{vmatrix} \tag{A.17}$$

or:

$$g^{23} = -\frac{\xi^3}{L^2} \frac{\partial lnh}{\partial \xi^2}. \tag{A.18}$$

The matrix in Eq. (2.5) gives the remaining components.