References

1. G. A. Baker, Jr. and T. A. Oliphant, Quart. Appl. Math. **17**, 361-373 (1960). References to germane literature will be found in this paper
2. G. Birkhoff and S. MacLane, *A survey of modern algebra*, Macmillan Co., 1941

HELICAL FLUID FLOWS*

By ROBERT H. WASSERMAN (*Michigan State University*)

Introduction. Potential helical flows have been completely described by G. Hamel [1]. Nemenyi and Prim, and N. Coburn have obtained some Beltrami helical flows [2, 3]. A simple description will be given here of all steady incompressible helical flows and all steady compressible helical flows with entropy constant along stream lines.

The equations of helical flow. The class of compressible flows to be considered here are those governed by the following differential equations in which v^* is the velocity, p is the pressure, ρ is the density, S is the entropy, and t^* is the unit tangent vector along the stream lines;

$$\nabla \cdot \rho v^* = 0 \text{ (continuity equation)} \tag{1}$$

$$(\rho v^* \cdot \nabla) v^* = - \nabla p \text{ (equation of motion)} \tag{2}$$

$$\rho = \rho(p, S) \text{ (equation of state)} \tag{3}$$

$$t^* \cdot \nabla S = 0 \text{ (entropy is constant along stream lines)} \tag{4}$$

For incompressible flows we must satisfy Eqs. (1) and (2) and the special case $\rho =$ constant of Eq. (3).

Now we assume that the flows are helical; i.e., the stream lines of the flow are parallel helices on coaxial circular cylinders. Such flows have the property $\nabla \cdot t^* = 0$. This may be seen immediately if we introduce cylindrical coordinates r, θ, z and decompose t^* according to

$$t^* = \sin \beta \theta^* + \cos \beta z^*.$$

Note that the angle β of the helices is in general a function of r.

With the condition $\nabla \cdot t^* = 0$ the continuity equation reduces to

$$t^* \cdot \nabla \ln \rho q = 0, \tag{5}$$

where q is the magnitude of the velocity. Also, from Eqs. (3) and (4)

$$t^* \cdot \nabla \rho - \frac{\partial \rho}{\partial p} t^* \cdot \nabla p = 0. \tag{6}$$

Finally, we write Eq. (2) in intrinsic form [4]:

$$t^* \cdot \nabla p = - \rho t^* \cdot \nabla \frac{q^2}{2}, \tag{7}$$

*Received November 10, 1958.

$$n^* \cdot \nabla p = -\rho q^2 \kappa, \tag{8}$$

$$b^* \cdot \nabla p = 0, \tag{9}$$

where n^* is the unit principal normal vector of the stream lines, b^* is the unit binormal vector of the stream lines, and κ is the curvature of the stream lines. Then Eqs. (6), (7), and (8) give us the result that either

(a) $$t^* \cdot \nabla q = t^* \cdot \nabla p = t^* \cdot \nabla \rho = 0$$

or

(b) $$q^2 = \frac{\partial p}{\partial \rho} \quad \text{and} \quad p = B(S) - \frac{A(S)}{\rho},$$

where the functions $A(S)$ and $B(S)$ are restricted only by the condition $A(S) > 0$. Clearly, the alternative (*b*), in which the equation of state includes that used by Chaplygin, and Kármán and Tsien, only occurs in the compressible case.

The fact that the helices are geodesics on the cylinders $r =$ const. means that n^* is normal to these cylinders. With this, and using Euler's equation for the normal curvature of a curve to evaluate κ, Eq. (8) becomes

$$\frac{\partial p}{\partial r} = \rho q^2 \frac{\sin^2 \beta}{r}. \tag{10}$$

The two classes of helical flows. In case (a) Eq. (10) reduces to

$$\frac{dp}{dr} = \rho q^2 \frac{\sin^2 \beta}{r}.$$

If in this equation it is understood that ρ and q are constant along stream lines (i.e., condition (a) holds), then it embodies all the conditions for helical flows in case (a). That is, there is a helical flow corresponding to any set of functions p, ρ, q, β satisfying this equation.

In case (b) if we introduce the coordinates

$$\alpha = z + \theta r \tan \beta,$$
$$\psi = z - \theta r \cot \beta,$$
$$r = r,$$

then Eq. (10) may be written

$$r \frac{\partial p}{\partial r} + (\alpha - \psi) \sin \beta \cos \beta \frac{d}{dr}(r \tan \beta) \frac{\partial p}{\partial \alpha} = (B - p) \sin^2 \beta.$$

If in this equation it is understood that p is only a function of r and α, and B is only a function of r and ψ then any solution gives a helical flow. As an example of a family of solutions we have those in which

$$B = \text{const.}$$

$$r \tan \beta = \lambda = \text{const.}$$

$$p = B - \frac{(\lambda^2 + r^2)^{1/2}}{r} F(\alpha),$$

where F is an arbitrary positive function of α.

Some general properties of flows of case (a). In case (a) which applies to all fluids except those having the special form of the equation of state of case (b) our basic equation is very simple and we can readily obtain some general properties of helical flows from it.

Thus we see that

I. p is a non-decreasing function of r, and it is easy to construct examples for which it is either bounded or unbounded.

II. ρ, q, and β may, in general, either increase or decrease with r. Moreover, ρ and q may vary from stream line to stream line on a cylinder as long as ρq^2 is constant on cylinders.

III. In certain important special cases the variation of ρ, q, and β is restricted. Thus, for example, in the incompressible case if the Bernoulli function is constant then q must decrease with r. In the compressible case, if the entropy and the stagnation enthalpy are constant, then q must decrease with r.

IV. For a polytropic gas our basic equation can be written quite simply in terms of the Mach number. We find that the Mach number may in general either increase or decrease with r. However, if the entropy and the stagnation enthalpy are constant then the Mach number must decrease with r.

V. With respect to the vorticity we note in particular

(i) in contrast to the other quantities of the flow [in case (a)] the vorticity *can* vary along the stream lines.

(ii) when the vorticity vector is normal to the stream lines the only possible stream line pattern is given by $\cot \beta / r =$ const. which is that of the simplest helical flow; that obtained by normal superposition of a potential vortex and a uniform rectilinear flow.

References

1. G. Hamel, *Potentialströmungen mit konstanter Geschwindigkeit,* Sitzungsber. Preuss. Akad. Wiss. Berlin, 1937
2. P. F. Nemenyi and R. C. Prim, *On the steady Beltrami flow of a perfect gas,* Proc. 7th Intl. Congr. Appl. Mechanics, vol. 2, London, 1948
3. N. Coburn, *Intrinsic form of the characteristic relations in the steady supersonic flow of a compressible fluid,* Quart. Appl. Math. 15, 237 (1957)
4. N. Coburn, *Intrinsic relations satisfied by the vorticity and velocity vectors in fluid flow theory,* Michigan Math. J. 1, 113 (1952)