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A MATHEMATICAL TREATMENT OF ONE-DIMENSIONAL SOIL CONSOLIDATION*

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Summary. Terzaghi's conception of the nature of one-dimensional soil consolidation [1] is shown to lead to a non-linear differential equation. A dimensional analysis of this equation and the boundary conditions of the standard consolidation test [2] gives a more general explanation of a well known linear relationship between the total consolidation $U(t)$ after a time t and $t^{1/2}$. By linearizing the equation in a general manner, an expression is obtained for $U(t)$ which includes secondary consolidation terms. Two solutions of the linearized equation are obtained; the first for the standard consolidation test and the second for consolidation under a boundary load increasing uniformly with time.

Introduction. A medium of clay saturated with water and subjected to a constant pressure, continues to contract in volume over a long period of time. Terzaghi, in his Erdbaumechanik [3], put forward a diffusion theory to explain the time dependent nature of this volume contraction. The following three ideas are the basis of his treatment and will be taken as the starting point of this paper.

(1) The medium is imagined as a porous water saturated skeleton of particles of negligible compressibility, so that a change in the volume of any region of the skeleton equals the volume of water displaced through its boundary.

(2) Stresses applied to the boundary of the clay produce a hydrostatic head h in the pore water and a stress at each point of the skeleton of such a direction and magnitude as to maintain static equilibrium at all instants of time.

(3) It is finally assumed that spatial variations in h cause the pore water to diffuse through the skeleton in accordance with Darcy's law, which equates the quantity of water flowing through any small plane surface σ in unit time to the product of the area of σ, the rate of change of h with distance along the normal to σ and a constant K called the permeability.

By considering the problem of the consolidation of clay in one direction only, the need for a tensor description of the stress distribution is avoided. The one-dimension problem is defined by the assumption that the fluid flow and the motion of the skeleton particles are in one direction only and that all relevant physical properties are constant at any instant of time on any plane with a normal along this direction. The stress function is then the pressure $P(z, t)$ which the skeleton in the neighborhood of the plane z can support at time t and, like K, will depend on the physical state of the solid matter

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of the clay. If the skeleton is a homogeneous aggregate of small non-spherical particles, the hydrological state of clay is described by two functions. The first, called the void ratio ϵ, and defined as the volume of water saturating unit volume of the porous structure in the neighborhood of the point, describes the degree of compaction of the aggregate, while the second, a distribution function ϕ, describes the orientation of the particles with respect to the direction of consolidation. It is evident that ϕ will depend on the previous history of the medium but it will be assumed to begin with that this is such that K can be treated as a function of ϵ only.

By assuming a linear relationship between ϵ and P and a constant value for K, Terzaghi derived an equation of the form

$$K \ \partial^2 P/\partial x^2 = \partial P/\partial t$$

for P.

Let us, however, derive the general mathematical formulation of the ideas outlined above and make approximations only when compelled to by the difficulty of solving specific problems.

The equation of one-dimensional consolidation theory. Let us define z to be the volume of the skeleton of the clay bounded by a cylinder S of unit cross-sectional area with its axis in the direction of consolidation, and the two planes 0, z fixed relative to the skeleton. If $L(z, t)$ is the distance between these two planes at time t,

$$\partial L/\partial z = 1 + \epsilon \quad \text{since} \quad L = \int_0^z (1 + \epsilon)\, dz.$$

The quantity of water $Q(t)$ in this region is given by the expression

$$Q = \int_0^z \epsilon \, dz \tag{1}$$

and hence the rate at which water flows through the planes 0, z into the region is $\partial Q/\partial t$.

Now by Darcy's law,

$$\frac{\partial Q}{\partial t} = \left[K \frac{\partial h}{\partial z} \Big/ \frac{\partial L}{\partial z} \right]_0^z \tag{2}$$

and so

$$\int_0^z \frac{\partial \epsilon}{\partial t} dz = \left[\frac{K}{1 + \epsilon} \frac{\partial h}{\partial z} \right]_0^z .$$

The derivative of this equation with respect to z is

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z} \left(\frac{K}{1 + \epsilon} \frac{\partial h}{\partial z} \right).$$

Suppose $P_0(t)$ is the pressure on two planes z_1, z_2, say, bounding the clay. Since consolidation is slow, the total internal pressure is also $P_0(t)$. Hence

$$P(z, t) + h(z, t) = P_0(t),$$

and the derivative of this equation with respect to z,

$$\partial P/\partial z = -\partial h/\partial z$$

inserted in (2) gives the one-dimensional consolidation equation in its final form;

$$\frac{\partial \epsilon}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{K}{1+\epsilon}\frac{\partial P}{\partial z}\right). \tag{3}$$

Two more equations connecting K, ϵ and P and a complete set of boundary conditions are needed to specify a soluble mathematical problem. The complexity of the geometry makes the problem of predicting the nature of these relationships a difficult one, and little can be said beyond the intuitively obvious statement that K tends to zero and P increases when ϵ decreases.

It is shown in the next section that, provided these two relationships do not explicitly involve z or t, the system can be solved for the case of the consolidation of a semi-infinite medium under a constant boundary pressure. In this case a dimensional analysis shows that the volume of water displaced from the skeleton is proportional to the square root of the duration of the consolidation.

The square root law. If the functions K and P are dependent on the void ratio ϵ alone, Eq. (3) is invariant under the family of similarity transformations

$$z \to \alpha z, \qquad t \to \alpha^2 t, \qquad \alpha > 0.$$

The boundary conditions associated with the problem of the consolidation of a semi-infinite medium under a constant boundary pressure P_0 are:

$$\epsilon = \epsilon_0 \text{ at } t = 0 \text{ for all } z > 0$$

and

$$P = P_0 \text{ at } z = 0 \text{ for all } t > 0.$$

These conditions are also invariant under each member of the set of transformations above. If the functions $K(\epsilon)$ and $P(\epsilon)$ are such that the boundary conditions uniquely determine the solution of (3), the functions $\epsilon(z, t)$ and $\epsilon(\alpha z, \alpha^2 t)$ must be identically equal for all $\alpha > 0$ in the region of positive values of z and t. This implied that ϵ is a function of $z/t^{1/2}$ only, for by writing z^* for αz and t^* for $\alpha^2 t$ and observing that $\epsilon(z^*, t^*)$ is independent of α for positive z and t, since in this region it is identically equal to $\epsilon(z, t)$, we obtain by differentiation, the equation

$$\partial\epsilon/\partial\alpha = z\,\partial\epsilon/\partial z^* + 2\alpha t\,\partial\epsilon/\partial t^* = 0.$$

This equation, when multiplied by α, gives the linear partial differential equation

$$z^*\,\partial\epsilon/\partial z^* + 2t^*\,\partial\epsilon/\partial t^* = 0$$

with the general solution

$$\epsilon = f(z^*/t^{*1/2}) = f(z/t^{1/2})$$

involving the arbitrary function f.

Let us write $y = z/t^{1/2}$, $\epsilon = f(y)$. Since

$$\partial\epsilon/\partial t = (\partial f/\partial y)(\partial y/\partial t) = -y/2t^{3/2}\cdot df/dy$$

and

$$\partial P/\partial z = 1/t^{1/2}\cdot dP/dy$$

Eq. (3) reduces to the ordinary differential equation

$$\frac{y}{2}\frac{df}{dy} = \frac{d}{dy}\left(\frac{K}{1+\epsilon}\frac{dP}{dy}\right) \tag{4}$$

while the boundary conditions become:

$$P = P_0 \text{ at } y = 0$$

and ϵ tends to ϵ_0 as y tends to infinity.

The progress of the consolidation is measured by the volume of water which has seeped through each unit of area of the surface $z = 0$ after the application of the boundary pressure P_0. The total consolidation $U(t)$ in this problem is the limiting value of $L(z, 0) - L(z, t)$ as z tends to infinity and hence

$$\begin{aligned} dU/dt &= \lim_{z\to\infty} \; -\partial L/\partial t \\ &= \lim_{z\to\infty} \; -\partial Q/\partial t \\ &= \left[\frac{K}{1+\epsilon}\frac{\partial P}{\partial z}\right]_{z=0} = \frac{K_0}{1+\epsilon_0}\left(\frac{dP}{dy}\right)_{y=0}\cdot\frac{1}{t^{1/2}} \end{aligned}$$

by a direct substitution from Eq. (2).

$K_0/(1 + \epsilon_0)$ depends only on the value ϵ_0 of ϵ which satisfies the boundary condition $P(\epsilon) = P_0$, while $(dP/dy)_{y=0}$ is a constant found by solving Eq. (4) and evaluating $dP/d\epsilon \cdot df/dy$ at $y = 0$. The differential equation for $U(t)$ can be easily integrated with the boundary condition $U = 0$ at $t = 0$ to give the result

$$U = 2At^{1/2}, \quad \text{where } A \text{ is the constant } \left[\frac{K}{1+\epsilon}\frac{dP}{dy}\right]_{y=0}. \tag{5}$$

This result is of interest since the semi-infinite problem is closely related to the well known consolidation problem with the modified boundary conditions

$$\epsilon = \epsilon_0 \text{ or } P = P_0' \equiv P(\epsilon_0) \text{ at } t = 0 \text{ for } 0 < z < z_1$$

and

$$P = P_0 \text{ for all } t > 0 \text{ on the planes } z = 0 \text{ and } z_1 .$$

During the initial stages of consolidation, in fact till $P(z_1/2t^{1/2})$ differs appreciably from P_0', this is essentially the superposition of two semi-infinite cases; and hence $U(t)$ is given initially by $4At^{1/2}$.

Terzaghi's linear theory of course, also predicts this result as it corresponds to a very special case of Eq. (3). Agreement between his formula for $U(t)$ and the experimental consolidation curves over the initial stages of compression has no bearing on the validity of the assumptions he has used in linearizing the theory. In many cases, agreement between his theory and experiment is in this region of linearity only, and the conclusion to be drawn is that in this region K and P are effectively functions of ϵ only.

The experimental determination of $K(\epsilon)$ and $P(\epsilon)$. If P is a function of ϵ only, there is no great difficulty in determining experimentally the functional relationship between them. This is usually done by carrying out a sequence of standard consolidation tests [2] and calculating the value of the void ratio at the conclusion of each test. The problem of determining the relationship between K and ϵ is a more difficult one.

The constant A in Eq. (5) seems at first to be proportional to $K(\epsilon_0)$. However, this is not so, because $(dP/dy)_{y=0}$ is indirectly related to the function $K(\epsilon)$ by the differential equation (4). This equation is insoluble except for simple forms of the functions f and k and these forms do not involve enough parameters to describe reasonably the functions. An empirical approach which partly avoids solving Eq. (4) is outlined below.

Suppose a sequence of standard consolidation tests is carried out on the same sample $0 < z < z_1$ and the load applied to the boundary is increased by a factor β at each step. If each experiment is carried through to the completion of the primary consolidation, and secondary consolidation is negligible, the boundary conditions for the rth test are;

$$P = \beta^{r-1} P_0 \text{ at } t = 0 \text{ for } 0 < z < z_1$$

$$P = \beta^r P_0 \text{ at } z = 0 \text{ and } z = z_1 \text{ for } t > 0.$$

The boundary conditions for the corresponding semi-infinite problem giving the gradient of the $(U, t^{1/2})$ line are

$$P \to \beta^{r-1} P_0 \text{ as } y \text{ tends to infinity and } P = \beta^r P_0 \text{ at } y = 0.$$

A measurement of ϵ at the conclusion of each test gives a set of points $(\epsilon_r, \beta^r P_0)$ which map the relationship between P and ϵ. Let us choose the values of the three constants C', C'', m in the empirical relation,

$$\epsilon = C'P^m + C'' \tag{6}$$

which gives the closest fit to the experimental curve.

If it is assumed that $K/(1 + \epsilon)$ can be approximated by an equation with two arbitrary constants C, n of the form

$$K/(1 + \epsilon) = CP^n, \tag{7}$$

the auxiliary equation for determining A_r, and so the gradient of the $(U, t^{1/2})$ curve for the rth test is

$$\frac{y}{2} \frac{d}{dy} (C'P^m) = \frac{d}{dy} \left(CP^n \frac{dP}{dy} \right). \tag{8}$$

Let $P = \beta^{r-1} p^*$ and $y = \beta^{(r-1)(n-m+1)/2} y'^*$ so that the equation determining A_r becomes

$$\frac{C'}{C} \frac{y^*}{2} \frac{dP^{*m}}{dy^*} = \frac{d}{dy^*} \left(P^{*n} \frac{dP^*}{dy^*} \right) \tag{9}$$

and the boundary conditions in terms of y and z' are

$$P^* \to P_0 \text{ as } y^* \text{ tends to infinity and } P^* = \beta P_0 \text{ at } y^* = 0.$$

This system is independent of r and so dP^*/dy^* has the same value at $y = 0$ for each value of r. We see from (5) the equation for A_r is

$$\begin{aligned} A_r &= \left[\frac{K}{1 + \epsilon} \frac{dP}{dP^*} \cdot \frac{dP^*}{dy^*} \frac{dy^*}{dy} \right]_{y=0} \\ &= CP_0^n \beta^{r(n+m+1)/2 + (n-m-1)/2} \left(\frac{dP^*}{dy^*} \right)_{y^*=0} \end{aligned} \tag{10}$$

or

$$\log A_r = \frac{r(n + m + 1)}{2} \log \beta + \log B,$$

where B is a constant independent of r. A graph of $\log A_r$ against $r \log \beta$ should therefore be a straight line of gradient $(n + m + 1)/2$ for the values of r over which the forms chosen for ϵ and $K/(1 + \epsilon)$ are valid approximations. C must now be found by solving Eq. (9) and it seems that this must be done numerically.

Linear theories of consolidation. The solution of a one-dimensional consolidation problem is determined theoretically by the experimental relationships connecting ϵ, P and K which give an explicit form to (3), and by the appropriate set of boundary conditions. In practice, mathematical difficulties make it necessary to linearize (3) in order to obtain analytical solutions for boundary conditions more complex than those considered above. This had always been done by assuming $K/(1 + \epsilon)$ is constant and choosing the linear relationship between P and ϵ which best fits the experimental curve.

Since there are two functions at our disposal, this linearizing process can be carried out in a less restrictive manner. It can readily be seen that to linearize (3), it is necessary that K, P and ϵ be related by an equation of the form

$$\frac{K}{1 + \epsilon} \frac{\partial P}{\partial z} = -k \frac{\partial \epsilon}{\partial z}, \tag{11}$$

where k is independent of ϵ. Equation (3) and this relationship, together give a linear equation for ϵ of the form

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z}\left(k \frac{\partial \epsilon}{\partial z}\right). \tag{12}$$

If k is constant or a function of t only, (12) can be put in the dimensionless form,

$$\frac{\partial \epsilon}{\partial \tau} = \frac{\partial^2 \epsilon}{\partial x^2}, \tag{13}$$

where $\tau = \int_0^t k/a^2\, dt$ and $x = z/a^2$, a being a convenient measure of length. τ will be used as a measure of time in all that follows and "a" will usually be half of the volume of the skeleton between the boundary planes and the cylinder S of unit cross-sectional area normal to these planes.

Equation (11) gives an explicit expression for K

$$K = -k \frac{\partial}{\partial P}\left(\epsilon + \frac{\epsilon^2}{2}\right) \tag{14}$$

which is fortunately not physically unreasonable. ϵ decreases monotonically to a finite limit as P increases and so K, as determined by (14), is positive for positive k and tends to zero as ϵ decreases. There is still freedom to choose ϵ as any function of P and t, as one more relationship between ϵ, K and P is required to specify the consolidation problem mathematically. It is interesting to note how previous consolidation theories comply with (14). Neglecting ϵ^2 and taking ϵ as a linear function of P, and k as a constant, gives a constant value for K as in Terzaghi's theory. The linear theory proposed by D. W. Taylor and W. Marchant in 1940 [4] has a constant K and an $\epsilon - P$ relationship of the form

$$\epsilon = \epsilon_0 - c(P - P_0) + (c - c')(P - P_0) \exp(-\mu t).$$

Equation (14) is satisfied by choosing k a function of t as defined by the equation

$$k(t)[c - (c - c') \exp(-\mu t)] = K$$

and a simple integration gives

$$\tau = \frac{K}{ca^2} \log \left[\frac{c \exp(\mu t) + (c' - c)}{c} \right].$$

Standard consolidation tests on many types of clays show that ϵ is a function of t as well as P, for, when subjected to the constant boundary pressure P_0, ϵ continues to decrease after the hydrostatic head h should have vanished. Equation (3) is still valid for a time-dependent $\epsilon - P$ relationship and so this behavior, usually described as secondary consolidation, can be incorporated in a linear theory by using a suitable $\epsilon - P - t$ relationship and a corresponding permeability K, defined by (14). The function $\epsilon(P, t)$ for many soils, can be adequately described by a form

$$\epsilon(P, t) = B - B' \log P - B'' \log(1 + \tau) \tag{15}$$

involving three constants B, B' and B''.

k can be chosen as either that constant which makes the curve

$$K = B'k[1 + B - B' \log P - B'' \log(1 + \tau)]/P$$

fit a known relationship between K and P closely, or that which gives the time scale $\tau = kt/a^2$, which agrees with an experimental curve.

The solution of the linear consolidation equation with general boundary conditions. Let us consider the one-dimensional consolidation of a homogeneous clay medium bounded by plane surfaces at $z = a$ and $-a$ and the cylinder of unit cross-sectional area, S, normal to these, under a boundary pressure $P_0(\tau)$, which varies during the experiment. We will also suppose there is an initial variation in the void ratio along the axis of the cylinder. In mathematical terms, we wish to find the solution of Eq. (13) subject to the boundary conditions

$$\epsilon = \epsilon_0(x) \text{ at } \tau = 0 \text{ for } x \text{ in the interval } -1 < x < 1$$

and

$$\epsilon = \epsilon[P_0(\tau), \tau] \equiv g(\tau) \text{ say, on } x = \pm 1 \text{ for } \tau > 0.$$

The Laplace transform of the solution is readily obtained by conventional methods and was found to be

$$\epsilon_L = (g_L + I_1) \frac{\cosh qx}{\cosh q} + I_2 \frac{\sinh qx}{\sinh q} - I_3,$$

where

$$I_1 = \int_0^1 \frac{\epsilon_0(x') + \epsilon_0(-x')}{2q} \sinh q(1 - x')\, dx',$$

$$I_2 = \int_0^1 \frac{\epsilon_0(x') - \epsilon_0(-x')}{2q} \sinh q(1 - x')\, dx',$$

$$I_3 = \int_0^1 \frac{\epsilon_0(x')}{q} \sinh q(x - x')\, dx',$$

$$\epsilon_L = \int_0^\infty e^{-s\tau}\epsilon(x, \tau)\, d\tau, \qquad g_L = \int_0^\infty e^{-s\tau} g(\tau)\, d\tau \quad \text{and} \quad q^2 = s.$$

The total consolidation is by definition $L(a, 0) - L(a, \tau) - L(-a, 0) + L(-a, \tau)$, where

$$L(z, \tau) = \int_0^z (1 + \epsilon)\, dz$$

and hence

$$U(\tau) = a \int_{-1}^1 (\epsilon_0 - \epsilon)\, dx$$

while

$$U_L \equiv \int_0^\infty e^{-s\tau} U\, d\tau = a \int_{-1}^1 \left(\frac{\epsilon_0}{s} - \epsilon_L\right) dx$$
$$= 2a\left[\int_0^1 \frac{\epsilon_0(x') + \epsilon_0(-x')}{2s} \frac{\cosh qx'}{\cosh q}\, dx' - g_L \frac{\tanh q}{q}\right]. \tag{16}$$

If $\epsilon_0(x)$ is a constant ϵ_0 say, plus an odd function, the first term of (16) simplifies and

$$U_L = 2a\left(\frac{\epsilon_0}{s} - g_L\right) \frac{\tanh q}{q}. \tag{17}$$

In general $\epsilon_0(x)$ affects the behavior of U near $\tau = 0$, especially if the deviation from a constant plus an odd function is greatest near $x = +$ or -1. In the standard consolidation test, this effect will appear as an initial departure from linearity in the $(U, t^{1/2})$ curve and so as a form of pre-primary consolidation.

The Laplace transform (17) can be readily inverted to give

$$U = 2a \int_0^\tau [\epsilon_0 - g(\tau - t)]\Theta_2(0 \mid i\pi t)\, dt, \tag{18}$$

where $\theta_2(\nu \mid t)$ is the theta function defined p. 388 of [5]. For large values of τ this integral is asymptotic to $2a(\epsilon_0 - g)$ while for small values of τ it behaves like

$$2a\pi^{-1/2} \int_0^\tau [\epsilon_0 - g(\tau - t)]t^{-1/2}\, dt \tag{19}$$

or like $4a\, [\epsilon_0 - g(0)]\, (\tau/\pi)^{1/2}$ if $dg/d\tau$ is small. Since $g(\tau) = \epsilon[P_0(\tau), \tau]$, $g(0)$ is the initial void ratio on the boundaries.

The standard consolidation test. The pressure on the boundaries in the standard consolidation test [2] is constant, P_0, say, and so $g(\tau) = \epsilon(P_0, \tau)$ is a function describing the secondary consolidation. If, for simplicity, we neglect any initial variation in the void ratio, we have the immediate result U behaves like $2a(\epsilon_0 - g)$ for large values of τ, which is for the relationship (15) $2aB'' \log (1 + \tau)$.

Let us consider in more detail, the consolidation of soils adequately described by Eq. (15). We see from Eq. (17) that the Laplace transform of the total consolidation is given by the expression

$$U_L = 2a\left[\frac{\epsilon_0 - \epsilon_1}{s} - \frac{B''}{s} e^s Ei(-s)\right]\frac{\tanh q}{q}, \tag{20}$$

where $\epsilon_1 = B - B' \log P_0$.

The first term of (20) is the transform of the well known solution of the standard consolidation problem, while the second gives the effect of secondary consolidation on U_L. These terms can be inverted by well known techniques of the Operational Calculus to give series of asymptotic solutions useful for small values of τ and others for large τ. Let us write U as $U_1 + U_2$, where U_1 is the primary and U_2 the secondary consolidation term.

For small values of τ

$$U_1 = 4a(\epsilon_0 - \epsilon_1)\left\{\pi^{-1/2}\tau^{1/2} + 2\sum_{r=1}^{\infty}(-1)^r[\pi^{-1/2}\tau^{1/2}\exp(-r^2/\tau) - r \operatorname{Erfc}(r/\tau^{1/2})]\right\} \tag{21}$$

and

$$U_2 = 8aB''\pi^{-1/2}\{(1+\tau)^{1/2}\log[\tau^{1/2} + (1+\tau)^{1/2}] - \tau^{1/2}\} + 4aB''\sum_{r=1}^{\infty}(-1)^r J_r, \tag{22}$$

where

$$J_r = \int_0^\tau \pi^{-1/2}t^{-1/2}\exp(-r^2/t)\log(1+\tau-t)\,dt.$$

As J_r is not expressible in finite terms, the series (22) is not very useful for values of τ for which J_1 cannot be neglected. However, we may use the first term till

$$2\log(1+\tau)\{\pi^{-1/2}\tau^{1/2}\exp(-1/\tau) - \operatorname{erfc}(1/\tau^{1/2})\}$$

which is an upper bound to J_1, ceases to be negligible. For large values of the dimensionless parameter τ

$$U_1 = 2a(\epsilon_0 - \epsilon_1)\left\{1 - \frac{8}{\pi^2}\sum_{r=0}^{\infty}\exp[-(2r+1)^2\pi^2\tau/4]/(2r+1)^2\right\}, \tag{23}$$

and

$$\begin{gathered} U_2 \sim 2aB''\left\{\log(1+\tau) + \sum_{r=0}^{\infty} M_r \exp[-\pi^2(2r+1)^2\tau/4] + N_r/(1+\tau)^{r+1}\right\}, \\ M_r = 8\pi^{-2}\exp[-\pi^2(2r+1)^2/4]Ei^*[\pi^2(2r+1)^2/4], \\ N_r = (-4)^{r+2}(4^{r+2} - 1)\, r\, !)B_{2r+4}/(2r+4)!, \end{gathered} \tag{24}$$

where B_{2r+4} are Bernoulli numbers and $Ei^*(x)$ is the logarithmic integral defined p. 2 of [6] as $li\,(e^x)$.

The first few terms of the asymptotic expression (24), which is useful for values of τ outside the range of (22), are

$$U_2 \sim 2aB''\left\{\log(1+\tau) - \frac{1}{3(1+\tau)} - \frac{2}{15(1+\tau)^2} - \cdots + .4755\exp(-2.4674\tau) + 18.895\exp(-22.207\tau) + \cdots\right\}.$$

Consolidation under a linearly increasing boundary pressure. It was pointed out earlier, that the standard consolidation test will not effectively justify approximations which are made to linearize Eq. (3). It is evident that in an experiment designed to test an approximation arising from (14), the pressure applied to the boundary planes must vary with time. Let us consider the simplest problem of this type, that with a linearly increasing boundary pressure.

Suppose a pressure P_0 is first applied to the boundaries and maintained constant until the void ratio attains a value ϵ_0 throughout the medium. Now, at $\tau = 0$ let us apply the boundary pressure

$$P_0(\tau) = P_0(1 + b\tau).$$

We see from (15) and (17) that

$$\begin{aligned} U_L &= -2a[B' \exp(s/b)Ei(-s/b) + B'' \exp(s)Ei(-s)] \frac{\tanh q}{sq} \\ &\equiv (U_3)_L + (U_2)_L \quad \text{say} \end{aligned} \tag{25}$$

U_2 is given directly by (22) and (24) and, by modifications in the derivation of these formulae, it can be shown that, for small values of τ,

$$U_3 = 8aB'(b\pi)^{-1/2}\{(1 + b\tau)^{1/2} \log [(b\tau)^{1/2} + (1 + b\tau)^{1/2}]\} + 4aB' \sum_{r=1}^{\infty} (-1)^r J'_r\,, \tag{26}$$

where

$$J'_r = \pi^{-1/2} \int_0^\tau t^{-1/2} \exp(-r^2/t) \log[1 + b(\tau - t)]\, dt$$

and J'_1 is bounded above by the expression

$$2 \log(1 + b\tau)[\pi^{-1/2}\tau^{1/2} \exp(-1/\tau) - \operatorname{erfc}(1/\tau^{1/2})],$$

while for large values of τ

$$U_3 \sim \frac{2aB'}{b} \log(1 + b\tau) + \sum_{r=0}^{\infty} M'_r \exp[-(2r + 1)^2\pi^2\tau/4] + N_r b^r/(1 + b\tau)^{r+1},$$

where N_r is as defined in (24) and

$$M'_r = 8\pi^{-2} \exp[-\pi^2(2r + 1)^2/4b]Ei^*[\pi^2(2r + 1)^2/4b]. \tag{27}$$

It will be found that the two approximate forms for U_2 and U_3 are unsatisfactory in an interval of values of τ. It would be useful to know J_1 and J'_1 in this interval and they could be calculated by a numerical integration of the integrals defining them, but it would be better in this case to use the appropriate form of (18).

The consolidation of media of compressible particles. It was assumed, in deriving Eq. (3), that the particles composing the skeleton of the clay had a negligible compressibility. This seems to offer one direction in which the theory presented in this paper can be generalized, but it is not difficult to introduce a compressibility factor into the treatment and show that the conclusions and formulae come out practically unchanged. If z is defined as the mass of the skeleton of the clay bounded by the cylinder S and the two planes 0 and z fixed relative to the skeleton and ϵ as the volume of water saturating unit mass of the porous structure, Eq. (1) is unchanged. If $L(z, t)$ is now defined by the equation

$$L = \int_0^z (\eta + \epsilon)\, dz,$$

where η is the volume of unit mass of the skeleton, Eq. (2) is also unchanged and Eq. (3) is replaced by

$$\frac{\partial \epsilon}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{K}{\eta + \epsilon}\frac{\partial P}{\partial z}\right). \tag{28}$$

The square root law is now valid if K, P and η can be expressed as functions of ϵ only. Finally, it can be seen that the linearizing approximation, (11), now becomes

$$\frac{K}{\eta + \epsilon}\frac{\partial P}{\partial z} = -k\frac{\partial \epsilon}{\partial z}. \tag{29}$$

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