TORSION AND EXTENSION OF HELICOIDAL SHELLS*

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1. Introduction. The present paper is concerned with the problem of rotationally symmetric deformations of thin elastic shells the middle surface of which is a portion of a right helicoid. Particular consideration is given to the problem of a uniformly pretwisted thin strip which is acted upon by tractions which result in equal and opposite axial forces F, and equal and opposite axial torques T (Fig. 1).

The differential equations for stresses and deformations of thin homogeneous, isotropic helicoidal shells, as used here, have been derived elsewhere [1]. In what follows they are employed in the form which they assume for rotationally symmetric states of stress and strain. Insofar as the problem of the pretwisted strip is concerned one of the essential aspects of the analysis is the connection between rotationally symmetric states of strain which depend on states of displacement which are not rotationally symmetric. The details of this connection are established in the present paper.

Of particular interest in the problem of the pretwisted strip are the relations between the applied force and torque on the one hand and the angle of elastic twist and the relative axial extension on the other hand. In this connection certain explicit results are presented which generalize earlier work of Chen Chu [2] regarding the torsional rigidity of the strip.

2. Equations for helicoidal shells. Let r, θ, z be cylindrical coordinates and let

$$z = a\theta \tag{2.1}$$

be the equation of the middle surface of the shell. The constant $2\pi a$ is the pitch of the helicoidal middle surface. The parametric curves r = constant and θ = constant on the middle surface of the shell form an orthogonal net but are not the lines of curvature.

The state of stress in the helicoidal shell (2.1) is described by stress resultants N_r, N_θ, $N_{r\theta}$, $N_{\theta r}$, Q_r and Q_θ and stress couples M_r, M_θ, $M_{r\theta}$ and $M_{\theta r}$ referred to tangential and normal directions at the edges of an element of the shell (Fig. 2). The differential equations of equilibrium of an element of the shell are [1],

$$\frac{\partial}{\partial r}(\alpha N_r) + \frac{\partial N_{\theta r}}{\partial \theta} - \frac{r}{\alpha} N_\theta + \frac{a}{\alpha} Q_\theta = 0, \tag{2.2}$$

$$\frac{\partial}{\partial r}(\alpha N_{r\theta}) + \frac{\partial N_\theta}{\partial \theta} + \frac{r}{\alpha} N_{\theta r} + \frac{a}{\alpha} Q_r = 0, \tag{2.3}$$

$$\frac{\partial}{\partial r}(\alpha Q_r) + \frac{\partial Q_\theta}{\partial \theta} - \frac{a}{\alpha}(N_{r\theta} + N_{\theta r}) = 0, \tag{2.4}$$

$$\frac{\partial}{\partial r}(\alpha M_r) + \frac{\partial M_{\theta r}}{\partial \theta} - \frac{r}{\alpha} M_\theta - \alpha Q_r = 0, \tag{2.5}$$

*Received August 4, 1958; typographically revised manuscript received December 29, 1958. A report on work carried out under the terms of an Office of Naval Research Contract at the Massachusetts Institute of Technology.

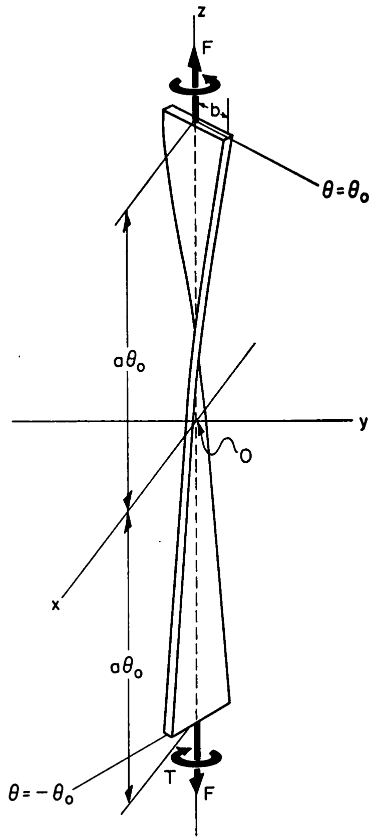

FIG. 1 Pretwisted strip

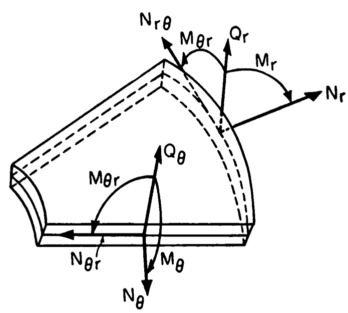

FIG. 2 Element of helicoidal shell

$$\frac{\partial}{\partial r}(\alpha M_{r\theta}) + \frac{\partial M_\theta}{\partial \theta} + \frac{r}{\alpha} M_{\theta r} - \alpha Q_\theta = 0, \tag{2.6}$$

$$N_{r\theta} - N_{\theta r} + \frac{a}{\alpha^2}(M_\theta - M_r) = 0. \tag{2.7}$$

The quantity α is defined by

$$\alpha = (a^2 + r^2)^{1/2}. \tag{2.8}$$

The system (2.2) to (2.7) is completed by a system of stress strain relations which here is taken in a form corresponding to the relations of Flügge [3] and Byrne [4] for lines of curvature coordinates and which is [1],

$$N_r = \frac{Eh}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) - \frac{a}{\alpha^2}\frac{Eh^3}{12(1-\nu^2)}\left[\tau^* - (1-\nu)\frac{a}{\alpha^2}(\epsilon_r - \epsilon_\theta)\right], \tag{2.9}$$

$$N_\theta = \frac{Eh}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r) - \frac{a}{\alpha^2}\frac{Eh^3}{12(1-\nu^2)}\left[\tau^* - (1-\nu)\frac{a}{\alpha^2}(\epsilon_\theta - \epsilon_r)\right], \tag{2.10}$$

$$N_{r\theta} = \frac{Eh}{2(1+\nu)}\gamma_{r\theta} - \frac{a}{\alpha^2}\frac{Eh^3}{24(1-\nu^2)}[(1+\nu)\kappa_r^* + (3-\nu)\kappa_\theta^*], \tag{2.11}$$

$$N_{\theta r} = \frac{Eh}{2(1+\nu)}\gamma_{r\theta} - \frac{a}{\alpha^2}\frac{Eh^3}{24(1-\nu^2)}[(1+\nu)\kappa_\theta^* + (3-\nu)\kappa_r^*], \tag{2.12}$$

$$M_r = \frac{Eh^3}{12(1-\nu^2)}\left(\kappa_r^* + \nu\kappa_\theta^* - \frac{1-\nu}{2}\frac{a}{\alpha^2}\gamma_{r\theta}\right), \tag{2.13}$$

$$M_\theta = \frac{Eh^3}{12(1-\nu^2)}\left(\kappa_\theta^* + \nu\kappa_r^* - \frac{1-\nu}{2}\frac{a}{\alpha^2}\gamma_{r\theta}\right), \tag{2.14}$$

$$M_{r\theta} = \frac{Eh^3}{12(1-\nu^2)}\left[\frac{1-\nu}{2}\tau^* - \frac{a}{\alpha^2}(\epsilon_\theta + \nu\epsilon_r)\right], \tag{2.15}$$

$$M_{\theta r} = \frac{Eh^3}{12(1-\nu^2)}\left[\frac{1-\nu}{2}\tau^* - \frac{a}{\alpha^2}(\epsilon_r + \nu\epsilon_\theta)\right]. \tag{2.16}$$

The strain quantities ϵ, γ, κ and τ are expressed in terms of *radial*, *circumferential* and *axial* components of middle surface displacement u, v and w as follows;

$$\epsilon_r = \frac{\partial u}{\partial r}, \tag{2.17}$$

$$\epsilon_\theta = \frac{r}{\alpha^2}\frac{\partial v}{\partial \theta} + \frac{a}{\alpha^2}\frac{\partial w}{\partial \theta} + \frac{r}{\alpha^2}u, \tag{2.18}$$

$$\gamma_{r\theta} = \frac{1}{\alpha}\frac{\partial u}{\partial \theta} + \frac{r}{\alpha}\frac{\partial v}{\partial r} + \frac{a}{\alpha}\frac{\partial w}{\partial r} - \frac{v}{\alpha}, \tag{2.19}$$

$$\kappa_r^* = -\frac{r}{\alpha}\frac{\partial^2 w}{\partial r^2} - \frac{a^2}{\alpha^3}\frac{\partial w}{\partial r} + \frac{a}{\alpha}\frac{\partial^2 v}{\partial r^2} - \frac{ar}{\alpha^3}\frac{\partial v}{\partial r} + \frac{a}{\alpha^3}v - \frac{a}{\alpha^3}\frac{\partial u}{\partial \theta}, \tag{2.20}$$

$$\kappa_\theta^* = -\frac{r}{\alpha^3}\frac{\partial^2 w}{\partial \theta^2} + \frac{a}{\alpha^3}\frac{\partial^2 v}{\partial \theta^2} - \frac{1}{\alpha}\frac{\partial w}{\partial r} + \frac{a}{\alpha^3}\frac{\partial u}{\partial \theta}, \tag{2.21}$$

$$\tau^* = -\frac{2r}{\alpha^2}\frac{\partial^2 w}{\partial r\,\partial \theta} + \frac{2a}{\alpha^2}\frac{\partial^2 v}{\partial r\,\partial \theta} + \frac{2(r^2 - a^2)}{\alpha^4}\frac{\partial w}{\partial \theta} - \frac{4ar}{\alpha^4}\frac{\partial v}{\partial \theta} - \frac{4ar}{\alpha^4}u. \tag{2.22}$$

For the formulation of boundary conditions it is necessary to have expressions for *effective* edge stress resultants which include the action of the edge twisting moments and their derivatives, insofar as they are statically equivalent to distributions of edge forces. *Radial*, *circumferential* and *axial* components of these effective edge stress resultants are, along edges $r =$ constant,

$$R_r = N_r + \frac{a}{\alpha^2}M_{r\theta}, \tag{2.23}$$

$$H_r = \frac{r}{\alpha} N_{r\theta} - \frac{a}{\alpha} Q_r - \frac{a}{\alpha^2} \frac{\partial M_{r\theta}}{\partial \theta}, \tag{2.24}$$

$$Z_r = \frac{a}{\alpha} N_{r\theta} + \frac{r}{\alpha} Q_r + \frac{r}{\alpha^2} \frac{\partial M_{r\theta}}{\partial \theta}, \tag{2.25}$$

and, along edges $\theta =$ constant,

$$R_\theta = N_\theta + \frac{a}{\alpha^2} M_{\theta r}, \tag{2.26}$$

$$H_\theta = \frac{r}{\alpha} N_\theta - \frac{a}{\alpha} Q_\theta - \frac{\partial}{\partial r}\left(\frac{a}{\alpha} M_{\theta r}\right), \tag{2.27}$$

$$Z_\theta = \frac{a}{\alpha} N_\theta + \frac{r}{\alpha} Q_\theta + \frac{\partial}{\partial r}\left(\frac{r}{\alpha} M_{\theta r}\right). \tag{2.28}$$

In addition this procedure introduces concentrated corner forces of magnitude $\pm (M_{\theta r} + M_{r\theta})$ in the direction of the normal to the middle surface of the shell, with components $\pm (r/\alpha)(M_{r\theta} + M_{\theta r})$ and $\mp (a/\alpha)(M_{r\theta} + M_{\theta r})$ in the axial and circumferential directions, respectively.

3. Boundary conditions for the problem of the pretwisted strip. We consider a helicoidal shell with edge coordinates $r = \pm b$ and $\theta = \pm \theta_0$. The usual polar coordinate interpretation is attached to the meaning of negative values of r; the point $(-r, \theta)$ is the image of the point (r, θ) under reflection in the z-axis.

We assume that the edges $r = \pm b$ are free of stress and that the edges $\theta = \pm \theta_0$ are acted upon by forces F and torques T in accordance with Fig. 1. We then have the following system of boundary conditions along edges $r =$ constant:

$$r = \pm b: \qquad R_r = H_r = M_r = Z_r = 0. \tag{3.1}$$

Along edges $\theta =$ constant the boundary conditions are taken in a form which insures that a rotationally symmetric state of stress will exist in the strip. Thus we prescribe, at $\theta = \pm \theta_0$:

$$\int_{-b}^{b} Z_\theta \, dr - \left[\frac{r}{\alpha}(M_{r\theta} + M_{\theta r})\right]_{-b}^{b} = F, \tag{3.2}$$

$$\int_{-b}^{b} H_\theta \, dr + \left[\frac{a}{\alpha}(M_{r\theta} + M_{\theta r})\right]_{-b}^{b} = 0, \tag{3.3}$$

$$\int_{-b}^{b} N_{\theta r} \, dr = 0, \tag{3.4}$$

$$\int_{-b}^{b} rH_\theta \, dr + \left[\frac{ar}{\alpha}(M_{r\theta} + M_{\theta r})\right]_{-b}^{b} = T, \tag{3.5}$$

$$\int_{-b}^{b} rZ_\theta \, dr - \left[\frac{r^2}{\alpha}(M_{r\theta} + M_{\theta r})\right]_{-b}^{b} = 0, \tag{3.6}$$

$$\int_{-b}^{b} M_\theta \, dr = 0. \tag{3.7}$$

4. Displacements and stresses for the pretwisted strip. Considerations of symmetry indicate that displacements for the pretwisted strip problem should be of the form

$u = u(r)$, $v = v_0(r)\theta$ and $w = w_0(r)\theta$. The requirement that the components of strain (2.17) to (2.22) be independent of θ leads to the conclusion that admissible displacements are of the form

$$u = u(r), \qquad v = k_1 r\theta, \qquad w = k_2\theta, \tag{4.1}$$

where k_1 and k_2 are constants.

For displacements of the form (4.1) we have further that $\kappa_r^* = \kappa_\theta^* = \gamma_{r\theta} = 0$. In view of Eqs. (2.11) to (2.14) this means that the displacements (4.1) are associated with the vanishing of four resultants and couples,

$$N_{r\theta} = N_{\theta r} = M_r = M_\theta = 0. \tag{4.2}$$

The equilibrium equation (2.3) then implies

$$Q_r = 0, \tag{4.3}$$

and the following system of differential equations for N_r, N_θ, $M_{r\theta}$, $M_{\theta r}$, Q_θ, and u remains

$$\alpha(\alpha N_r)' - rN_\theta + aQ_\theta = 0, \tag{4.4}$$

$$\alpha(\alpha M_{r\theta})' + rM_{\theta r} - \alpha^2 Q_\theta = 0, \tag{4.5}$$

$$N_r = \frac{Eh}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) - \frac{a}{\alpha^2}\frac{Eh^3}{12(1-\nu^2)}\left[\tau^* - (1-\nu)\frac{a}{\alpha^2}(\epsilon_r - \epsilon_\theta)\right], \tag{4.6}$$

$$N_\theta = \frac{Eh}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r) - \frac{a}{\alpha^2}\frac{Eh^3}{12(1-\nu^2)}\left[\tau^* - (1-\nu)\frac{a}{\alpha^2}(\epsilon_\theta - \epsilon_r)\right], \tag{4.7}$$

$$M_{r\theta} = \frac{Eh^3}{12(1-\nu^2)}\left[\frac{1-\nu}{2}\tau^* - \frac{a}{\alpha^2}(\epsilon_\theta + \nu\epsilon_r)\right], \tag{4.8}$$

$$M_{\theta r} = \frac{Eh^3}{12(1-\nu^2)}\left[\frac{1-\nu}{2}\tau^* - \frac{a}{\alpha^2}(\epsilon_r + \nu\epsilon_\theta)\right]. \tag{4.9}$$

In these equations primes denote differentiation with respect to r, and

$$\epsilon_r = u', \qquad \epsilon_\theta = \frac{r}{\alpha^2}u + k_1\frac{r^2}{\alpha^2} + k_2\frac{a}{\alpha^2}, \tag{4.10}$$

$$\tau^* = -\frac{4ar}{\alpha^4}u + k_1\frac{2a(a^2 - r^2)}{\alpha^4} + k_2\frac{2(r^2 - a^2)}{\alpha^4}. \tag{4.11}$$

Three of the four boundary conditions (3.1) are automatically satisfied and the fourth may be written in the form

$$r = \pm b: \qquad \alpha^2 N_r + aM_{r\theta} = 0. \tag{4.12}$$

Of the six boundary conditions (3.2) to (3.7) four are automatically satisfied and the remaining two may be simplified (upon elimination of Q_θ) by suitable integration by parts to read

$$F = \int_{-b}^{b}\left(\frac{a}{\alpha}N_\theta + \frac{r^2 - a^2}{\alpha^3}M_{r\theta} + \frac{r^2}{\alpha^3}M_{\theta r}\right)dr, \tag{4.13}$$

$$T = \int_{-b}^{b}\left(\frac{r^2}{\alpha}N_\theta + \frac{a^3 - ar^2}{\alpha^3}M_{r\theta} + \frac{a^3}{\alpha^3}M_{\theta r}\right)dr. \tag{4.14}$$

5. Non-dimensionalization and simplification. In Eq. (4.1) we may write

$$k_1 = a\omega, \qquad k_2 = a\delta, \tag{5.1}$$

where ω and δ are respectively the angle of twist and the axial extension, both per unit of axial length.

We further introduce a dimensionless displacement and dimensionless resultants and couples as follows:

$$u_0 = \frac{u}{b}, \tag{5.2}$$

$$n_r = \frac{N_r}{Eh}, \qquad n_\theta = \frac{N_\theta}{Eh}, \qquad q_\theta = \frac{b^2}{h^2}\frac{Q_\theta}{Eh}, \tag{5.3}$$

$$m_{r\theta} = \frac{bM_{r\theta}}{Eh^3}, \qquad m_{\theta r} = \frac{bM_{\theta r}}{Eh^3}. \tag{5.4}$$

The quantities u_0 , n, q, and m are considered as functions of a dimensionless coordinate ρ defined by

$$\rho = \frac{r}{b}. \tag{5.5}$$

We set finally

$$\lambda = \frac{b}{a}. \tag{5.6}$$

The parameter λ measures the pretwist of the strip and vanishes for an untwisted plate located in the xz-plane.

Introduction of (5.1) to (5.6) into Eqs. (4.4) to (4.11) transforms these equations to the following form

$$\frac{d}{d\rho}\left[(1 + \lambda^2\rho^2)^{1/2} n_r\right] - \frac{\lambda^2\rho}{(1 + \lambda^2\rho^2)^{1/2}} n_\theta + \frac{1}{(1 + \lambda^2\rho^2)^{1/2}}\frac{h^2}{b^2} q_\theta = 0, \tag{5.7}$$

$$\frac{d}{d\rho}\left[(1 + \lambda^2\rho^2)^{1/2} m_{r\theta}\right] + \frac{\lambda^2\rho}{(1 + \lambda^2\rho^2)^{1/2}} m_{\theta r} - (1 + \lambda^2\rho^2)^{1/2} q_\theta = 0, \tag{5.8}$$

$$(1 - \nu^2)n_r = \epsilon_r + \nu\epsilon_\theta - \frac{1}{12}\frac{h^2}{b^2}\frac{\lambda}{1 + \lambda^2\rho^2}\left[b\tau^* - \frac{(1 - \nu)\lambda}{1 + \lambda^2\rho^2}(\epsilon_r - \epsilon_\theta)\right], \tag{5.9}$$

$$(1 - \nu^2)n_\theta = \epsilon_\theta + \nu\epsilon_r - \frac{1}{12}\frac{h^2}{b^2}\frac{\lambda}{1 + \lambda^2\rho^2}\left[b\tau^* - \frac{(1 - \nu)\lambda}{1 + \lambda^2\rho^2}(\epsilon_\theta - \epsilon_r)\right], \tag{5.10}$$

$$12(1 - \nu^2)m_{r\theta} = \frac{1 - \nu}{2} b\tau^* - \frac{\lambda}{1 + \lambda^2\rho^2}(\epsilon_\theta + \nu\epsilon_r), \tag{5.11}$$

$$12(1 - \nu^2)m_{\theta r} = \frac{1 - \nu}{2} b\tau^* - \frac{\lambda}{1 + \lambda^2\rho^2}(\epsilon_r + \nu\epsilon_\theta), \tag{5.12}$$

where

$$\epsilon_r = \frac{du_0}{d\rho}, \qquad \epsilon_\theta = \frac{\lambda^2\rho}{1 + \lambda^2\rho^2} u_0(\rho) + \frac{\lambda\rho^2 b\omega}{1 + \lambda^2\rho^2} + \frac{\delta}{1 + \lambda^2\rho^2}, \tag{5.13}$$

$$b\tau^* = \frac{-4\lambda^3\rho}{(1+\lambda^2\rho^2)^2}u_0 + 2b\omega\frac{1-\lambda^2\rho^2}{(1+\lambda^2\rho^2)^2} + 2\delta\lambda\frac{\lambda^2\rho^2-1}{(1+\lambda^2\rho^2)^2}. \tag{5.14}$$

The system (5.7) to (5.14) is to be solved subject to the boundary conditions (4.12), which take the dimensionless form

$$\rho = \pm 1: \qquad n_r + \frac{\lambda}{1+\lambda^2\rho^2}\frac{h^2}{b^2}m_{r\theta} = 0. \tag{5.15}$$

The expressions (4.13) and (4.14) for the force F and the torque T assume the form

$$F = Ehb\int_{-1}^{1}\left[\frac{n_\theta}{(1+\lambda^2\rho^2)^{1/2}} + \frac{\lambda^3\rho^2-\lambda}{(1+\lambda^2\rho^2)^{3/2}}\frac{h^2}{b^2}m_{r\theta} + \frac{\lambda^3\rho^2}{(1+\lambda^2\rho^2)^{3/2}}\frac{h^2}{b^2}m_{\theta r}\right]d\rho, \tag{5.16}$$

$$T = Ehb^2\int_{-1}^{1}\left[\frac{\lambda\rho^2}{(1+\lambda^2\rho^2)^{1/2}}n_\theta + \frac{1-\lambda^2\rho^2}{(1+\lambda^2\rho^2)^{3/2}}\frac{h^2}{b^2}m_{r\theta} + \frac{1}{(1+\lambda^2\rho^2)^{3/2}}\frac{h^2}{b^2}m_{\theta r}\right]d\rho. \tag{5.17}$$

We now assume that the pretwist parameter λ is of order of magnitude unity and not large compared with unity, and that all dimensionless resultants and couples are of the same order of magnitude. Considering the fact that $h^2/b^2 \ll 1$, we may then neglect certain of the terms in the system (5.7) to (5.17) and thus reduce it to the form

$$\frac{d}{d\rho}[(1+\lambda^2\rho^2)^{1/2}n_r] - \frac{\lambda^2\rho}{(1+\lambda^2\rho^2)^{1/2}}n_\theta = 0, \tag{5.18}$$

$$\frac{d}{d\rho}[(1+\lambda^2\rho^2)^{1/2}m_{r\theta}] + \frac{\lambda^2\rho}{(1+\lambda^2\rho^2)^{1/2}}m_{\theta r} = (1+\lambda^2\rho^2)^{1/2}q_\theta, \tag{5.19}$$

$$n_r = \frac{\epsilon_r + \nu\epsilon_\theta}{1-\nu^2}, \tag{5.20}$$

$$n_\theta = \frac{\epsilon_\theta + \nu\epsilon_r}{1-\nu^2}, \tag{5.21}$$

$$m_{r\theta} = \frac{1}{12(1-\nu^2)}\left[\frac{1-\nu}{2}b\tau^* - \frac{\lambda}{1+\lambda^2\rho^2}(\epsilon_\theta + \nu\epsilon_r)\right], \tag{5.22}$$

$$m_{\theta r} = \frac{1}{12(1-\nu^2)}\left[\frac{1-\nu}{2}b\tau^* - \frac{\lambda}{1+\lambda^2\rho^2}(\epsilon_r + \nu\epsilon_\theta)\right], \tag{5.23}$$

$$\rho = \pm 1: \qquad n_r = 0, \tag{5.24}$$

$$F = Ehb\int_{-1}^{1}\frac{n_\theta}{(1+\lambda^2\rho^2)^{1/2}}d\rho, \tag{5.25}$$

$$T = Ehb^2\int_{-1}^{1}\left[\frac{\lambda\rho^2}{(1+\lambda^2\rho^2)^{1/2}}n_\theta + \frac{1-\lambda^2\rho^2}{(1+\lambda^2\rho^2)^{3/2}}\frac{h^2}{b^2}m_{r\theta} + \frac{1}{(1+\lambda^2\rho^2)^{3/2}}\frac{h^2}{b^2}m_{\theta r}\right]d\rho. \tag{5.26}$$

6. Reduction of the differential equations. The resultants n_r, n_θ, and the couples $m_{r\theta}$ and $m_{\theta r}$ are expressed in terms of $u_0(\rho)$ by (5.20) to (5.23). The moment equilibrium equation (5.19) serves to express q_θ in terms of u_0 :

$$12(1 - \nu^2)q_\theta = \frac{-\nu\lambda}{1 + \lambda^2\rho^2}\frac{d^2u_0}{d\rho^2} - \frac{(4 - 3\nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^2}\frac{du_0}{d\rho} + b\omega\frac{(1 - \nu)\lambda^4\rho^3 - 2(3 - \nu)\lambda^2\rho}{(1 + \lambda^2\rho^2)^3} + \delta\frac{(7 - 5\nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^3}. \tag{6.1}$$

The remaining condition (5.18) for radial force equilibrium provides the following differential equation for u_0 .

$$\frac{d^2u_0}{d\rho^2} + \frac{\lambda^2\rho}{1 + \lambda^2\rho^2}\frac{du_0}{d\rho} + \frac{\nu\lambda^2 - \lambda^4\rho^2}{(1 + \lambda^2\rho^2)^2}u_0 = b\omega\left[\frac{(1 - \nu)\lambda\rho}{1 + \lambda^2\rho^2} - \frac{(1 + \nu)\lambda\rho}{(1 + \lambda^2\rho^2)^2}\right] + \delta\frac{(1 + \nu)\lambda^2\rho}{(1 + \lambda^2\rho^2)^2}. \tag{6.2}$$

The boundary condition (5.24) then becomes

$$\rho = \pm 1: \qquad \frac{du_0}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2}u_0 + b\omega\frac{\nu\lambda\rho^2}{1 + \lambda^2\rho^2} + \delta\frac{\nu}{1 + \lambda^2\rho^2} = 0. \tag{6.3}$$

Thus $u_0(\rho)$ is a solution of the boundary value problem (6.2) and (6.3); it will depend on ω and δ.

In view of the linearity of the boundary value problem we may write

$$u_0 = b\omega u_1 + \delta u_2 \tag{6.4}$$

and split (6.2), (6.3) into the following two boundary value problems for u_1 and u_2 .

$$Lu_1 = \frac{(1 - \nu)\lambda\rho}{1 + \lambda^2\rho^2} - \frac{(1 + \nu)\lambda\rho}{(1 + \lambda^2\rho^2)^2}, \tag{6.5}$$

$$\rho = \pm 1: \qquad \frac{du_1}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2}u_1 = \frac{-\nu\lambda}{1 + \lambda^2}, \tag{6.6}$$

$$Lu_2 = \frac{(1 + \nu)\lambda^2\rho}{(1 + \lambda^2\rho^2)^2}, \tag{6.7}$$

$$\rho = \pm 1: \qquad \frac{du_2}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2}u_2 = \frac{-\nu}{1 + \lambda^2\rho^2}, \tag{6.8}$$

where L is the differential operator on the left side of (6.2).

The differential equation $Lu = 0$ is, in different notation, the equation derived by Sanders [5] for helicoidal shell problems in which the displacements are independent of θ. It is possible to reduce this equation to hypergeometric form in a number of different ways, but no use will be made of this possibility in what follows.

7. Influence coefficients. When the boundary value problems (6.5) to (6.8) have been solved, we shall have all quantities expressed in terms of known functions of ρ, and the constants ω and δ. Thus by (5.20) to (5.23) and (6.4),

$$(1 - \nu^2)n_r = b\omega\left[\frac{du_1}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2}u_1 + \frac{\nu\lambda\rho^2}{1 + \lambda^2\rho^2}\right] + \delta\left[\frac{du_2}{d\rho} + \frac{\nu\lambda^2\rho}{1 + \lambda^2\rho^2}u_2 + \frac{\nu}{1 + \lambda^2\rho^2}\right], \tag{7.1}$$

$$(1 - \nu^2)n_\theta = b\omega\left[\nu \frac{du_1}{d\rho} + \frac{\lambda^2\rho}{1 + \lambda^2\rho^2} u_1 + \frac{\lambda\rho^2}{1 + \lambda^2\rho^2}\right] + \delta\left[\nu \frac{du_2}{d\rho} + \frac{\lambda^2\rho}{1 + \lambda^2\rho^2} u_2 + \frac{1}{1 + \lambda^2\rho^2}\right], \tag{7.2}$$

$$12(1 - \nu^2)m_{r\theta} = b\omega\left[\frac{-\nu\lambda}{1 + \lambda^2\rho^2}\frac{du_1}{d\rho} - \frac{(3 - 2\nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^2} u_1 + \frac{(1 - \nu) - (2 - \nu)\lambda^2\rho^2}{(1 + \lambda^2\rho^2)^2}\right] + \delta\left[\frac{-\nu\lambda}{1 + \lambda^2\rho^2}\frac{du_2}{d\rho} - \frac{(3 - 2\nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^2} u_2 + \frac{(1 - \nu)\lambda^3\rho^2 - (2 - \nu)\lambda}{(1 + \lambda^2\rho^2)^2}\right], \tag{7.3}$$

$$12(1 - \nu^2)m_{\theta r} = b\omega\left[\frac{-\lambda}{1 + \lambda^2\rho^2}\frac{du_1}{d\rho} - \frac{(2 - \nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^2} u_1 + \frac{(1 - \nu) - \lambda^2\rho^2}{(1 + \lambda^2\rho^2)^2}\right] + \delta\left[\frac{-\lambda}{1 + \lambda^2\rho^2}\frac{du_2}{d\rho} - \frac{(2 - \nu)\lambda^3\rho}{(1 + \lambda^2\rho^2)^2} u_2 + \frac{(1 - \nu)\lambda^3\rho^2 - \lambda}{(1 + \lambda^2\rho^2)^2}\right]. \tag{7.4}$$

When these relations are introduced into the force and torque conditions (5.25), (5.26) we have two relations between the force F and torque T on the one hand and the angle of twist ω and the axial extension δ on the other hand. These relations may be written in the form

$$F = (2Ehb\gamma_{F\delta})\,\delta + (\tfrac{2}{3}Ehb^2\lambda\gamma_{F\omega})\omega, \tag{7.5}$$

$$T = (\tfrac{2}{3}Ehb^2\lambda\gamma_{T\delta})\,\delta + (\tfrac{2}{5}Ehb^3\lambda^2\gamma_{T\omega} + \tfrac{2}{3}Gh^3b\gamma)\omega, \tag{7.6}$$

where the dimensionless influence coefficients γ are given by

$$\gamma_{F\delta} = \frac{1}{1 - \nu^2}\int_0^1 \left[\frac{\nu}{(1 + \lambda^2\rho^2)^{1/2}}\frac{du_2}{d\rho} + \frac{\lambda^2\rho u_2}{(1 + \lambda^2\rho^2)^{3/2}} + \frac{1}{(1 + \lambda^2\rho^2)^{3/2}}\right] d\rho, \tag{7.7}$$

$$\gamma_{F\omega} = \frac{3}{1 - \nu^2}\frac{1}{\lambda}\int_0^1 \left[\frac{\nu}{(1 + \lambda^2\rho^2)^{1/2}}\frac{du_1}{d\rho} + \frac{\lambda^2\rho}{(1 + \lambda^2\rho^2)^{3/2}} u_1 + \frac{\lambda\rho^2}{(1 + \lambda^2\rho^2)^{3/2}}\right] d\rho, \tag{7.8}$$

$$\gamma_{T\delta} = \frac{3}{1 - \nu^2}\int_0^1 \left[\frac{\nu\rho^2}{(1 + \lambda^2\rho^2)^{1/2}}\frac{du_2}{d\rho} + \frac{\lambda^2\rho^3}{(1 + \lambda^2\rho^2)^{3/2}} u_2 + \frac{\rho^2}{(1 + \lambda^2\rho^2)^{3/2}}\right] d\rho, \tag{7.9}$$

$$\gamma_{T\omega} = \frac{5}{1 - \nu^2}\frac{1}{\lambda}\int_0^1 \left[\frac{\nu\rho^2}{(1 + \lambda^2\rho^2)^{1/2}}\frac{du_1}{d\rho} + \frac{\lambda^2\rho^3}{(1 + \lambda^2\rho^2)^{3/2}} u_1 + \frac{\lambda\rho^4}{(1 + \lambda^2\rho^2)^{3/2}}\right] d\rho, \tag{7.10}$$

and finally

$$\gamma = \int_0^1 \frac{d\rho}{(1 + \lambda^2\rho^2)^{7/2}} = \frac{1 + (4\lambda^2/3) + (8\lambda^4/15)}{(1 + \lambda^2)^{5/2}}. \tag{7.11}$$

In the integrals (7.7) to (7.11) use has been made of the fact that u_1 and u_2 are odd in ρ, so that the integrands are even and $\int_{-1}^{1} = 2\int_0^1$.

It will be shown in Sec. 8 that when $\lambda = 0$ we have $\gamma_{F\delta} = \gamma_{F\omega} = \gamma_{T\delta} = \gamma_{T\omega} = \gamma = 1$, so that (7.5) and (7.6) become

$$F = 2Ehb, \qquad T = \tfrac{2}{3}Gh^3b\omega,$$

corresponding to well-known results in the theory of extension and torsion of a flat plate by end loads.

A straightforward application of Green's formula, making use of the fact that u_1

and u_2 are solutions of (6.5) to (6.8), shows that $\gamma_{F\omega} = \gamma_{T\delta}$, as is in fact required by the reciprocal work theorem of elasticity.

In addition to the flexibility relations (7.5), (7.6), we have the associated inverse equations

$$\left.\begin{aligned} \omega &= K_{\omega T}T + K_{\omega F}F \\ \delta &= K_{\delta T}T + K_{\delta F}F \end{aligned}\right\}. \tag{7.12}$$

It is readily shown from (7.5) and (7.6) that

$$\left.\begin{aligned} K_{\omega T} &= \frac{3}{2Gh^3b[\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2 f(\lambda)]} \\ K_{\delta T} = K_{\omega F} &= \frac{-\lambda k(\lambda)}{2Gh^3[\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2 f(\lambda)]} \\ K_{\delta F} &= \frac{k_{\delta F}(\lambda) + (3Eb^2/5Gh^2)\lambda^2 g(\lambda)}{2Ehb[\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2 f(\lambda)]} \end{aligned}\right\}, \tag{7.13}$$

where the abbreviations

$$\left.\begin{aligned} f(\lambda) &= \frac{9}{4}\gamma_{T\omega} - \frac{5}{4}\frac{\gamma_{F\omega}\gamma_{T\delta}}{\gamma_{F\delta}}, \\ k(\lambda) &= \frac{\gamma_{F\omega}}{\gamma_{F\delta}} = \frac{\gamma_{T\delta}}{\gamma_{F\delta}}, \\ k_{\delta F}(\lambda) &= \frac{\gamma}{\gamma_{F\delta}}, \qquad g(\lambda) = \frac{\gamma_{T\omega}}{\gamma_{F\delta}} \end{aligned}\right\} \tag{7.14}$$

have been used.

In the absence of axial forces, the torque T and twist ω are related by

$$T = I\omega,$$

where the torsional rigidity I is derived from (7.12) and (7.13) as

$$I = \frac{1}{K_{\omega T}} = \tfrac{2}{3}Gh^3b\left[\gamma(\lambda) + \frac{4}{15}\frac{E}{G}\frac{b^2}{h^2}\lambda^2 f(\lambda)\right]. \tag{7.15}$$

In the absence of torque the axial force F and the relative extension δ are related by

$$F = K\delta,$$

where the axial stiffness K is given by

$$K = \frac{1}{K_{\delta F}} = 2Ehb\,\frac{\gamma(\lambda) + (4Eb^2/15Gh^2)\lambda^2 f(\lambda)}{k_{\delta F}(\lambda) + (3Eb^2/5Gh^2)\lambda^2 g(\lambda)}. \tag{7.16}$$

In the following section the first three terms in the power series expansions for $\gamma_{F\delta}(\lambda)$, $\gamma_{F\omega}(\lambda)$, $\gamma_{T\omega}(\lambda)$, $\gamma(\lambda)$, $f(\lambda)$, $k(\lambda)$, $k_{\delta F}(\lambda)$, and $g(\lambda)$ are obtained by perturbation methods.

8. Perturbation solutions. All quantities of physical interest have been expressed in terms of the solutions of the two boundary value problems (6.5) to (6.8). Inspection of these problems indicates that $u_1(\rho, \lambda)$ is odd in λ and $u_2(\rho, \lambda)$ is even in λ, while both are odd functions of ρ. For sufficiently small λ, we obtain solutions in the form

$$\left.\begin{aligned} u_1(\rho, \lambda) &= \lambda u_1^{(1)}(\rho) + \lambda^3 u_1^{(3)}(\rho) + \cdots, \\ u_2(\rho, \lambda) &= u_2^{(0)}(\rho) + \lambda^2 u_2^{(2)}(\rho) + \cdots. \end{aligned}\right\} \tag{8.1}$$

Introduction of these assumptions into the boundary value problems (6.5) to (6.8) leads to a sequence of boundary value problems for the $u_i^{(j)}(\rho)$. These may be solved successively by repeated integration. The results of such calculations are

$$\begin{aligned} u_1 = {} & -\frac{\nu}{3}\rho^3\lambda + \left[-\frac{1-\nu^2}{4}\rho + \left(\frac{1}{20} + \frac{\nu}{5} + \frac{\nu^2}{60}\right)\rho^5\right]\lambda^3 \\ & + \left[(1-\nu^2)\frac{9+5\nu}{36}\rho + \frac{(1-\nu^2)(1+\nu)}{24}\rho^3 - \frac{(135 + 383\nu + 57\nu^2 + \nu^3)}{2520}\rho^7\right]\lambda^5 \\ & + O(\lambda^7), \end{aligned} \tag{8.2}$$

$$\begin{aligned} u_2 = {} & -\nu\rho + \left[-\frac{1-\nu^2}{2}\rho + \frac{(1+\nu)^2}{6}\rho^3\right]\lambda^2 + \left[\frac{3}{8}(1+\nu)(1-\nu^2)\rho \right. \\ & \left. + \frac{1}{12}(1-\nu^2)(1+\nu)\rho^3 - \frac{(1+\nu)^2(15+\nu)}{120}\rho^5\right]\lambda^4 \\ & + \left[\frac{(1+\nu)^2(1-\nu)(31-37\nu)}{144}\rho - \frac{(1-\nu^2)(1+\nu)^2}{16}\rho^3 \right. \\ & \left. - \frac{(1-\nu^2)(1+\nu)(15+\nu)}{240}\rho^5 + \frac{(1-\nu^2)(515 + 60\nu + \nu^2)}{5040}\rho^7\right]\lambda^6 \\ & + O(\lambda^8). \end{aligned} \tag{8.3}$$

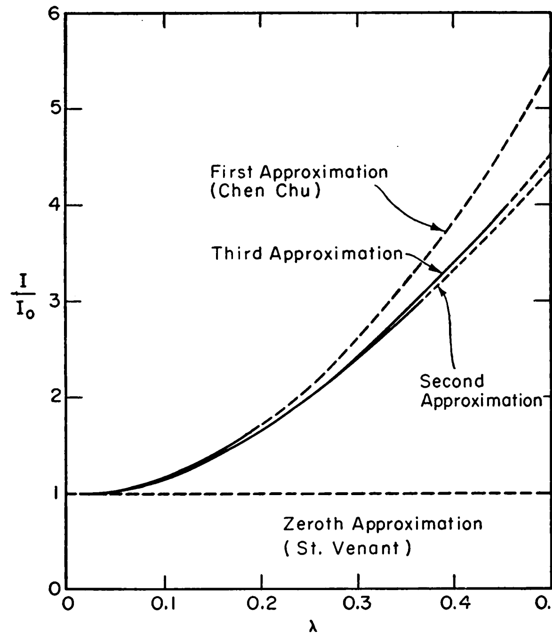

FIG. 3 Torsional rigidity ratio *vs.* pretwist parameter for $\nu = 1/3$ and $2b/h = 10$

Introduction of the perturbation solutions (8.2) and (8.3) into the integrals (7.7) to (7.10) gives the power series expansions in λ of $\gamma_{F\delta}$, $\gamma_{F\omega} = \gamma_{T\delta}$, and $\gamma_{T\omega}$; $\gamma(\lambda)$ may be directly expanded from (7.11). These expansions are

$$\left.\begin{aligned}
\gamma_{F\delta} &= 1 - \frac{3 + 4\nu}{6}\lambda^2 + \frac{29 + 88\nu + 56\nu^2}{120}\lambda^4 + O(\lambda^6),\\
\gamma_{F\omega} = \gamma_{T\delta} &= 1 - \frac{9 + 8\nu}{10}\lambda^2 + \frac{161 + 304\nu + 152\nu^2}{280}\lambda^4 + O(\lambda^6),\\
\gamma_{T\omega} &= 1 - \frac{45 + 20\nu}{42}\lambda^2 + \frac{531 + 504\nu + 232\nu^2}{648}\lambda^4 + O(\lambda^6),\\
\gamma &= 1 - \frac{7}{6}\lambda^2 + \frac{63}{40}\lambda^4 + O(\lambda^6).
\end{aligned}\right\} \tag{8.4}$$

These in turn provide the expansions of $f(\lambda)$, $k(\lambda)$, $k_{\delta F}(\lambda)$ and $g(\lambda)$ defined in (7.14).

$$\left.\begin{aligned}
f &= 1 - \frac{33 - 4\nu}{42}\lambda^2 + \frac{427 - 152\nu + 8\nu^2}{840}\lambda^4 + O(\lambda^6),\\
k &= 1 - \frac{6 + 2\nu}{15}\lambda^2 + \frac{42 + 6\nu - 4\nu^2}{315}\lambda^4 + O(\lambda^6),\\
k_{\delta F} &= 1 - \frac{2}{3}(1 - \nu)\lambda^2 + \frac{45 - 38\nu - \nu^2}{45}\lambda^4 + O(\lambda^6),\\
g &= 1 - \frac{12 - 4\nu}{21}\lambda^2 + \frac{828 - 486\nu + 52\nu^2}{2835}\lambda^4 + O(\lambda^6).
\end{aligned}\right\} \tag{8.5}$$

In particular the torsional rigidity I of formula (7.15) takes the form

$$\begin{aligned}
I = I_0\bigg[&\left(1 - \frac{7}{6}\lambda^2 + \frac{63}{40}\lambda^4 + \cdots\right)\\
&+ \frac{4}{15}\frac{E}{G}\frac{b^2}{h^2}\lambda^2\left(1 - \frac{33 - 4\nu}{42}\lambda^2 + \frac{427 - 152\nu + 8\nu^2}{840}\lambda^4 + \cdots\right)\bigg],
\end{aligned} \tag{8.6}$$

where

$$I_0 = I_{\lambda=0} = \tfrac{2}{3}Gh^3b \tag{8.7}$$

is the St. Venant torsional rigidity of a flat plate of thickness h and width $2b$. If only the first term in each power series in parenthesis in (8.6) is retained, we obtain the Chen Chu approximation [2]

$$I \cong I_0\left(1 + \frac{4}{15}\frac{E}{G}\frac{b^2}{h^2}\lambda^2\right) \tag{8.8}$$

indicating the increase in torsional stiffness for small pretwist. Figure 3 compares the Chu approximation (8.8) with the second approximation (retaining λ^2 terms in the two power series) and the third approximation (retaining λ^4 terms).

Corresponding results for the axial stiffness K of (7.16) are obtained by inserting the power series for γ, f, $k_{\delta F}$ and g into (7.16). There follows

$$\frac{K}{K_0} = \left[\left(1 - \frac{7}{6}\lambda^2 + \frac{63}{40}\lambda^4 + \cdots\right) + \frac{4}{15}\frac{E}{G}\frac{b^2}{h^2}\lambda^2\left(1 - \frac{33 - 4\nu}{42}\lambda^2 + \frac{427 - 152\nu + 8\nu^2}{840}\lambda^4 + \cdots\right)\right]\left[\left(1 - \frac{2 - 2\nu}{3}\lambda^2 + \frac{45 - 38\nu - \nu^2}{45}\lambda^4 + \cdots\right) + \frac{3}{5}\frac{E}{G}\frac{b^2}{h^2}\lambda^2\left(1 - \frac{12 - 4\nu}{21}\lambda^2 + \frac{828 - 486\nu + 52\nu^2}{2835}\lambda^4 + \cdots\right)\right]^{-1}, \tag{8.9}$$

where

$$K_0 = K_{\lambda=0} = 2Ehb \tag{8.10}$$

is the axial stiffness of a flat plate according to plane stress. If only the first term is retained in each of the four power series in parentheses in (8.9) there follows what may be considered as an analogue of the Chu approximation;

$$K \cong K_0 \frac{1 + (4Eb^2/15Gh^2)\lambda^2}{1 + (3Eb^2/5Gh^2)\lambda^2}. \tag{8.11}$$

Figure 4 compares the first approximation (8.11) with the second and third approximations obtained by retaining the λ^2 and λ^4 terms, respectively, in the four power series in parentheses in (8.9).

In addition to the influence coefficients, the stress resultants and couples themselves may be calculated by introducing the expansions (8.2) and (8.3) into the expression

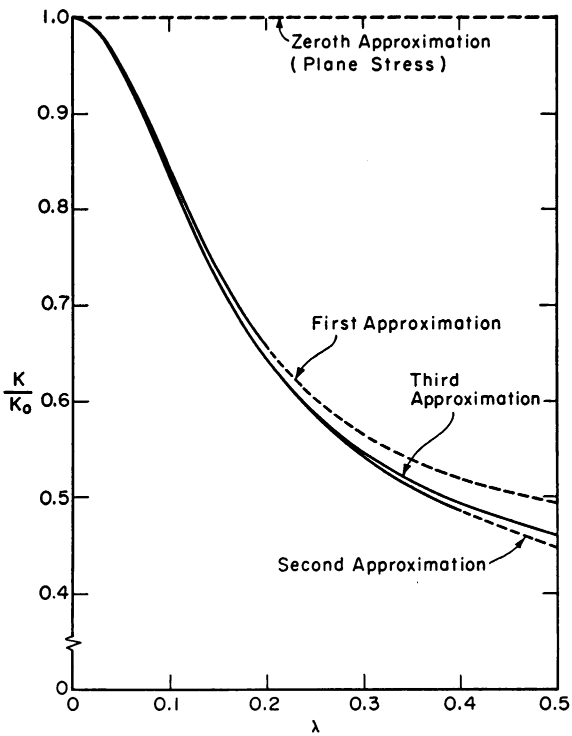

FIG. 4 Extensional stiffness ratio *vs.* pretwist parameter for $\nu = 1/3$ and $2b/h = 10$

(7.1) to (7.4) for n_r, n_θ, $m_{r\theta}$, and $m_{\theta r}$. We obtain

$$n_r = b\omega\left[-\tfrac{1}{4}(1-\rho^4)\lambda^3 + \left(\frac{9+5\nu}{36} + \frac{1-\nu}{8}\rho^2 - \frac{27+\nu}{72}\rho^6\right)\lambda^5 + \cdots\right]$$
$$+ \delta\left[-\tfrac{1}{2}(1-\rho^2)\lambda^2 + \left(\frac{3+3\nu}{8} + \frac{1-\nu}{4}\rho^2 - \frac{5+\nu}{8}\rho^4\right)\lambda^4 + \cdots\right], \tag{8.12}$$

$$n_\theta = b\omega\left[\rho^2\lambda - \left(\frac{\nu}{4} + \frac{12+\nu}{12}\rho^4\right)\lambda^3 + \left(\frac{9\nu+5\nu^2}{36} - \frac{2-\nu-\nu^2}{8}\rho^2\right.\right.$$
$$\left.\left. + \frac{378+57\nu+\nu^2}{360}\rho^6\right)\lambda^5 + \cdots\right] + \delta\left[1 - \left(\frac{\nu}{2} + \frac{2+\nu}{2}\rho^2\right)\lambda^2\right.$$
$$\left. + \left(\frac{3\nu+3\nu^2}{8} - \frac{2-\nu-\nu^2}{4}\rho^2 + \frac{28+17\nu+\nu^2}{24}\rho^4\right)\lambda^4 + \cdots\right], \tag{8.13}$$

$$12(1+\nu)m_{r\theta} = b\omega\left[1 - (4+\nu)\rho^2\lambda^2 + \left(\frac{\nu+\nu^2}{4} + \frac{84+33\nu+\nu^2}{12}\rho^4\right)\lambda^4 + \cdots\right]$$
$$+ \delta\left[-(2+\nu)\lambda + \left(\frac{\nu+\nu^2}{2} + \frac{10+9\nu+\nu^2}{2}\rho^2\right)\lambda^3 + \left(-\frac{3\nu+6\nu^2+3\nu^3}{8}\right.\right.$$
$$\left.\left. + \frac{6-\nu-8\nu^2-\nu^3}{4}\rho^2 - \frac{204+217\nu+38\nu^2+\nu^3}{24}\rho^4\right)\lambda^5 + \cdots\right], \tag{8.14}$$

$$12(1+\nu)m_{\theta r} = b\omega\left[1 - 3\rho^2\lambda^2 + \left(\frac{1+\nu}{4} + \frac{57+5\nu}{12}\rho^4\right)\lambda^4 + \cdots\right]$$
$$+ \delta\left[-\lambda + \left(\frac{1+\nu}{2} + \frac{5+3\nu}{2}\rho^2\right)\lambda^3 + \left(-\frac{3+6\nu+3\nu^2}{8}\right.\right.$$
$$\left.\left. + \frac{1-2\nu-3\nu^2}{4}\rho^2 - \frac{101+82\nu+5\nu^2}{24}\rho^4\right)\lambda^5 + \cdots\right]. \tag{8.15}$$

If (7.12), (7.13) and (8.5) are used to express ω and δ in terms of F and T in (8.12) to (8.15), the expression for stress resultants and couples are obtained in terms of F and T. The corresponding expressions are lengthy and will not be given here.

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