ON PARAMOUNT MATRICES*

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Introduction. A paramount matrix is a symmetric matrix of real numbers such that any principal minor must be at least as large as the absolute value of any other minor of the same order built from the same rows. In this paper a new property of paramount matrices is cited and proved, and some remarks are made relevant to the problem of the synthesis of multiport resistive networks. Unfortunately, to achieve a concise and rigorous presentation, a rather formal mathematical notation was developed, and some of the results, although not particularly profound, may be obscured by the mathematical formalism. Hence, a long introduction is included which describes the results in less exact but more comprehensible terms.

After several paragraphs introducing the notation, it is shown in Theorem 9 that if any elements in the main diagonal of a paramount matrix are increased and all other elements remain fixed, the resulting new matrix is still paramount. The remainder of the paper is devoted to the problem of the synthesis of multiport resistive networks.

It is well known [1] that if A is either the open-circuit impedance matrix or short circuit admittance matrix of a multiport resistive network, then A is paramount. However, it is not known whether paramountcy is sufficient for the realization of such a network. More precisely, if A is an arbitrary** paramount matrix of order m, it is not known whether there exists an m-port resistive network such that A is either the open-circuit impedance matrix or short-circuit admittance matrix of this network. Tellegen and Elias [2] have shown that the preceding statement is true if $m \leq 3$, but when m is arbitrary, the answer to the question remains one of the leading unsolved problems. This paper does not attempt to answer the question for an arbitrary m, but it does indicate a technique suggested by the the preceding result which may ultimately help to solve the problem.

Observe that an arbitrary mth order symmetric paramount matrix A has $m(m + 1)/2$ independent elements. Thus, an m-port resistive network whose open-circuit impedance matrix or short-circuit admittance matrix is A can be expected to contain, in general, at least $m(m + 1)/2$ resistive branches.

Note, however, that by Theorem 9, the set of all real numbers x such that a paramount matrix results when the element A_{11} of A is replaced by x, is a semi-infinite closed interval bounded on the left. Let b be the least element of this interval and let B be the matrix obtained from A when A_{11} is replaced by b. Now, if there exists a resistive network whose open-circuit impedance matrix is B, it is reasonable to expect that there exists such a network with $(m(m + 1)/2) - 1$ resistive branches since the number of independent elements of B is $(m(m + 1)/2) - 1$. Furthermore, if such a network is available and a resistance of $A_{11} - b$ ohms is introduced in series with port 1, then a resistive

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**An arbitrary paramount matrix of order m is a paramount matrix with exactly $m(m + 1)/2$ independent elements.

network is obtained whose open-circuit impedance matrix is A. Thus, to realize A, we need consider only geometrical structures realizing B, with $(m(m + 1)/2) - 1$ resistive branches. This idea can be generalized as follows.

In this paper, in Sec. 17, a number $n(A)$ is defined for any paramount matrix A. This number $n(A)$ is the maximum number such that the realization of A can be achieved by combining $n(A)$ resistances in series or parallel with the ports of a resistive network of

$$\frac{[\text{order }(A)][1 + \text{order }(A)]}{2} - n(A)$$

resistive branches that realizes a certain minimal paramount matrix obtained from A. This minimal paramount matrix obtained from A is minimal in the sense that it is irreducible as defined in Sec. 14. Thus, realization of A has been simplified to the consideration of the realization of such a minimal paramount matrix by a resistive structure with only

$$\frac{[\text{order }(A)][1 + \text{order }(A)]}{2} - n(A)$$

resistive branches. This represents a simpler geometrical realization problem, and because of the maximality of $n(A)$, it is the best simplification which can be effected by utilizing the concept described above.

It is easy to show that for any arbitrary third-order paramount matrix A, $n(A) = 3$. Thus as shown by Elias and Telegen [2], any arbitrary third-order paramount matrix A is always realizable as the open-circuit impedance matrix of a resistive network of six branches which has three resistive branches in series or parallel with the ports and three more resistive branches realizing the minimal* paramount matrix obtained from A. The same geometrical structure will always produce a resistive three-port network whose open-circuit impedance matrix is equal to an arbitrary third-order paramount matrix A.

Unfortunately, the situation is more complicated in the case of an arbitrary fourth-order paramount matrix. It is not true that for each arbitrary fourth-order paramount matrix A, the number $n(A)$ is the same. It is easy to exhibit such a matrix A with $n(A) = 2$ and another such matrix A with $n(A) = 4$. Clearly the minimal paramount matrix resulting in the former case will have eight independent elements, while the minimal paramount matrix resulting in the latter case will have six independent elements. This leads the writer to conjecture that if paramountcy is indeed a sufficient condition for the realization of a fourth-order matrix as the open-circuit impedance matrix of a resistive network, the same geometrical structure will not suffice for each arbitrary fourth-order paramount matrix A but will depend upon the number $n(A)$. The formal mathematics follows.

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*Consider the subnetwork of Tellegen's network which does not contain any of the three resistances in series or in parallel with the ports; agree to call any three-branch resistive three-port network a minimal resistive three-port, if and only if it has the geometrical configuration of this subnetwork. It was observed by Prof. R. M. Foster of the Polytechnic Institute of Brooklyn that a third-order paramount matrix is realizable as the open-circuit impedance matrix of a minimal resistive three-port network, if and only if the matrix is irreducible (as defined in Sec. 14).

1. Definitions.

(i) $R = \{x \mid x \text{ is a real number}\}$

(ii) $\omega = \{x \mid x \text{ is a positive integer}\}$

(iii) $\operatorname{dmn} f = \{x \mid \text{for some } y, (x, y) \varepsilon f\}$

(iv) $\operatorname{rng} f = \{x \mid \text{for some } y, (y, x) \varepsilon f\}$

(v) $f \mid A = f \cap \{(x, y) \mid x \varepsilon A\}$

(vi) If $x \varepsilon \omega$, then

$$\langle x \rangle = \omega \cap \{y \mid y \leq x\}$$

(vii) pA = the cardinal number of A.

2. Definitions.

(i) For each $x \varepsilon \omega$
$S^x = \{u \mid u \text{ is an increasing sequence with } 1 < pu \text{ and } \operatorname{rng} u \subset \langle x \rangle\}$,

(ii) For each $x \varepsilon \omega$ and $i \varepsilon \langle x \rangle$

$$S_i^x = S^x \cap \{u \mid i \not\varepsilon \operatorname{rng} u\}.$$

3. Definitions.

(i) $M = \{A \mid A \text{ is a function, } \operatorname{dmn} A = (\langle m \rangle \times \langle m \rangle) \text{ for some } m \varepsilon \omega \text{ and } \operatorname{rng} A \subset R\}$.

(ii) If $A \varepsilon M$ and $(i, j) \varepsilon \operatorname{dmn} A$,

$$A_{ij} = A(i, j).$$

(iii) If $A \varepsilon M$, order $(A) = (p \operatorname{dmn} A)^{1/2}$.

(iv) $K = M \cap \{A \mid A_{ji} = A_{ij} \text{ for } (i, j) \varepsilon \operatorname{dmn} A \text{ and } 2 \leq \text{order } (A)\}$.

(v) For $m \varepsilon \omega$,

$$M^m = M \cap \{A \mid m = \text{order } (A)\}.$$

(vi) If $A \varepsilon M$, det (A) is the customary determinant of the square matrix A.

(vii) If $A \varepsilon M$ and det $(A) \neq 0$, then A^{-1} is the customary inverse of A.

4. Definitions.

If $u \varepsilon S^x$,

$$[u * x]$$

is that increasing sequence of positive integers such that

$$\operatorname{dmn}[u * x] = \langle x - pu \rangle$$

$$[u * x]_1 = \inf (\langle x \rangle - \operatorname{rng} u)$$

$$[u * x]_i = \inf (\langle x \rangle - \operatorname{rng} u - \operatorname{rng}([u * x] \mid \langle i - 1 \rangle))$$

$$\text{for } i \varepsilon \langle x - pu \rangle \quad \text{and} \quad i \neq 1.$$

5. Definitions.

(i) For $A \varepsilon M$

$$G(A) = (S^{\text{order}(A)} \times S^{\text{order}(A)}) \cap \{(u, v) \mid pu = pv\}.$$

(ii) For $A \varepsilon M$ and $i \varepsilon \langle \text{order } (A) \rangle$

$$G_i(A) = G(A) \cap (S_i^{\text{order}(A)} \times S^{\text{order}(A)}).$$

6. Definitions.

For $A \varepsilon M$ and $(u, v) \varepsilon G(A)$

$$A_v^u \text{ is that element of } M^{\text{order}(A)-pu}$$

such that

$$(A_v^u)_{ij} = A([u * \text{order}(A)]_i, [v * \text{order}(A)]_j)$$

for

$$(i, j) \varepsilon (\langle \text{order}(A) - pu \rangle \times \langle \text{order}(A) - pu \rangle).$$

7. Definition.

A is paramount if and only if $A \varepsilon K$ and

$$\det(A_u^u) - |\det(A_v^u)| \geq 0$$

whenever $(u, v) \varepsilon G(A)$.

8. Definition.

If $A \varepsilon M$, $x \varepsilon R$ and $i \varepsilon \langle \text{order}(A) \rangle$, then

$$[A, i, x]$$

is that element of $M^{\text{order}(A)}$ such that

$$[A, i, x]_{jk} = A_{jk}$$

if $(j, k) \varepsilon \operatorname{dmn} A$ and $(j, k) \neq (i, i)$.

$$[A, i, x]_{ii} = x.$$

9. Theorem.

Suppose that A is paramount and that $i \varepsilon \langle \text{order}(A) \rangle$. Let

$$B = \{x | [A, i, x] \text{ is paramount}\}.$$

Then B is a closed, semi-infinite interval, bounded on the left.
Proof.
Let $0 < t \varepsilon R$. It suffices to show that $[A, i, A_{ii} + t]$ is paramount. Pick any $(u, v) \varepsilon G_i([A, i, A_{ii} + t])$.
It suffices to show that

$$\det([A, i, A_{ii} + t]_u^u) \geq |\det([A, i, A_{ii} + t]_v^u)|.$$

Suppose first that

$$pu = pv = \text{order}(A) - 1.$$

In this case

$$\det([A, i, A_{ii} + t]_u^u) = A_{ii} + t.$$

Also, if $i \not\varepsilon \operatorname{rng} v$, then

$$\det([A, i, A_{ii} + t]^u_v) = A_{ii} + t;$$

while if $i \in \operatorname{rng} v$, then

$$\det([A, i, A_{ii} + t]^u_v) = A_{ij}$$

for some $j \in \langle \text{order}(A) \rangle$ with $j \neq i$, and paramountcy of A implies the desired inequality.

Thus, assume now that

$$pu = pv < \text{order}(A) - 1.$$

Since $u \in S_i^{\text{order}(A)}$, $i \in \operatorname{rng}[u * \text{order}(A)]$. Let $\bar{\imath} \in \operatorname{dmn}[u * \text{order}(A)]$. such that

$$[u * \text{order}(A)]_{\bar{\imath}} = i.$$

Suppose now that $i \notin \operatorname{rng} v$.

Let $\bar{\jmath} \in \operatorname{dmn}[v * \text{order}(A)]$ such that $[v * \text{order}(A)]_{\bar{\jmath}} = i$. Then

$$\det([A, i, A_{ii} + t]^u_v) = -1^{\bar{\imath}+\bar{\jmath}} t \det(([A, i, A_{ii} + t]^u_v)^{\{(1,\bar{\imath})\}}_{\{(1,\bar{\jmath})\}}) + \det(A^u_v).$$

Clearly there exists $(\bar{u}, \bar{v}) \in G(A)$ such that

$$([A, i, A_{ii} + t]^u_v)^{\{(1,\bar{\imath})\}}_{\{(1,\bar{\jmath})\}} = A^{\bar{u}}_{\bar{v}}.$$

Thus,

$$\det([A, i, A_{ii} + t]^u_v) = -1^{\bar{\imath}+\bar{\jmath}} t \det(A^{\bar{u}}_{\bar{v}}) + \det(A^u_v).$$

Note that the above equation is valid when $u = v$ and $\bar{\imath} = \bar{\jmath}$. Hence, the fact that A is paramount implies

$$\begin{aligned} \det([A, i, A_{ii} + t]^u_u) - |\det([A, i, A_{ii} + t]^u_v)| & \\ \geq t \det(A^{\bar{u}}_{\bar{u}}) + \det(A^u_u) - (t\,|\det(A^{\bar{u}}_{\bar{v}})| + |\det(A^u_v)|) & \\ = t(\det(A^{\bar{u}}_{\bar{u}}) - |\det(A^{\bar{u}}_{\bar{v}})|) + (\det(A^u_v) - |\det(A^u_v)|) \geq 0. & \end{aligned}$$

Finally, if $i \in \operatorname{rng} v$, then

$$\det([A, i, A_{ii} + t]^u_v) = \det(A^u_v),$$

and

$$\begin{aligned} \det([A, i, A_{ii} + t]^u_u) - |\det([A, i, A_{ii} + t]^u_v)| & \\ = t \det(A^{\bar{u}}_{\bar{u}}) + (\det(A^u_u) - |\det(A^u_v)|) \geq 0. & \end{aligned}$$

The proof is complete.

10. Definition.

B is a reduction of A if and only if A is paramount, and

$$B = [A, i, \inf\{t \mid [A, i, t] \text{ is paramount}\}]$$

for some $i \in \langle \text{order}(A) \rangle$ such that

$$A_{ii} > \inf\{t \mid [A, i, t] \text{ is paramount}\}.$$

11. Definition.

P is a reduction sequence of A if and only if A is paramount, and P is such a sequence that

$$P_1 = A$$

and

$$P_i \text{ is a reduction of } P_{i-1} \text{ for } 1 < i \in \text{dmn } P.$$

12. Remark.

It is clear from Definitions 10 and 11 that if P is a reduction sequence of A, then

$$pP \leq \text{order } (A).$$

13. Remark.

Suppose that T is such a sequence that

(i) For $i \in \text{dmn } T$ $T(i)$ is a sequence to M.
(ii) $T(1)$ is a reduction sequence of A.
(iii) $(T(i))_{pT(i)}$ is non singular for each $i \in \text{dmn } T$.
(iv) $T(i + 1)$ is a reduction sequence of $[(T(i))_{pT(i)}]^{-1}$ for each $i \in \text{dmn } T$ such that $(i + 1) \in \text{dmn } T$.

Then, clearly, by Jacobi's theorem, whenever $i \in \text{dmn } T$ and $(i + 2) \in \text{dmn } T$,

$$pT(i + 2) < pT(i) \leq \text{order } (A).$$

Thus a sequence T satisfying the above conditions must be finite.

14. Definition.

A is irreducible if and only if A is paramount and there exists no B such that B is a reduction of A.

15. Definition.

T reduces A completely if and only if A is paramount and T is such a finite sequence that

(i) For $i \in \text{dmn } T$ $T(i)$ is a sequence to M.
(ii) $T(1)$ is a reduction sequence of A.
(iii) If $1 < pT$ then $(T(i))_{pT(i)}$ is non singular for $i \in \langle pT - 1\rangle$.
(iv) If $1 < pT$ then $T(i + 1)$ is a reduction sequence of $[(T(i))_{pT(i)}]^{-1}$ for $i \in \langle pT - 1\rangle$.
(v) $(T(pT)_{pT(pT)}$ is irreducible and singular, or, $1 < pT$, and $(T(pT))_{pT(pT)}$ and $(T(pT - 1))_{pT(pT-1)}$ are both irreducible.

16. Remark.

If T reduces A completely, then T is maximal in the following precise sense.

Theorem.

Let T reduce A completely and let U reduce A completely such that $T \subset U$. Then $T = U$.

17. Definitions.

(i) If T reduces A completely,

$$n(T, A) = \sum_{i \in \text{dmn } T} pT(i).$$

(ii) If A is paramount and A is not irreducible,

$$n(A) = \sup \{n(T, A) \mid T \text{ reduces } A \text{ completely}\}.$$

If A is paramount and irreducible $n(A) = 0$.

18. Remark.

It is not known whether the following statement is true:
if T reduces A completely and U reduces A completely, then

$$n(T, A) = n(U, A).$$

19. Remark.

If A is paramount, the number $n(A)$ defined in 17 (*ii*) may be relevant to the problem of the realization of A as either the open-circuit impedance matrix or the short-circuit admittance matrix of a resistive network. This topic is discussed in the Introduction. Clearly, if 18 is true, the computation of $n(A)$ will be greatly simplified, and thus determination of the validity of 18 may be of help in the ultimate solution of the realization problem.

20. Acknowledgment.

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References

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