QUASI-TRIDIAGONAL MATRICES AND TYPE-INSENSITIVE DIFFERENCE EQUATIONS*

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1. Introduction. In solving linear partial differential equations by finite difference methods a boundary value problem is reduced to solving a set of linear equations. In such instances the matrix involved usually takes a special form and consists mainly of zeros. Many of these matrices fall into the class to be considered here which may be called *quasi-tridiagonal* matrices. That is, we consider partitioned matrices of the form

$$Q = \begin{bmatrix} M_1 & E_1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ D_2 & M_2 & E_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & D_3 & M_3 & E_3 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & D_{q-1} & M_{q-1} & E_{q-1} \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & D_q & M_q \end{bmatrix} = [D_n\,, M_n\,, E_n]_1^q\,, \tag{1.1}$$

where the D_n, M_n, E_n are matrices with the same number of rows, E_n, M_{n+1}, D_{n+2} have the same number of columns, and the M_n are square. We propose to solve

$$Qv = g \tag{1.2}$$

by direct methods.

Various discretizations lead to matrices of this type. That the usual finite-difference approximation of certain boundary problems for the Poisson and the bi-harmonic equation yield matrices of the form (1.1) has been shown by O. Karlqvist [4], A. F. Cornock [2], L. H. Thomas [8] and others [6, 7]. Whereas the usual method of solution of the resulting linear equation is by iteration these authors propose direct methods for solving the above problems.

The processes described here are, in part, extensions of those described by Karlqvist [4] and Cornock [2]. Furthermore it is shown that the finite difference equations obtained from symmetric positive systems, as defined by K. O. Friedrichs [5], also fall into the class of matrices of the form (1.1). These include, in addition to pure elliptic or hyperbolic equations, a certain class of boundary problems for equations of mixed type such as the Tricomi equation. Where iterative methods for such problems seem to present difficulties, even when Q is positive definite and symmetric, the direct methods for solving (1.2) are shown below to be feasible for this larger class of problems.

A criterion is given for the process to apply which is similar to that found for the

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LDU theorem [3]. It is also shown that when the M_n have the same order p, and the D_n are easily invertible, then the process may be reduced to multiplication of matrices and the inversion of one matrix of order p. This last fact was noted by Cornock [2] for the Poisson and bi-harmonic case.

More generally, if for $r = 1, 2, \cdots, k$; $k \leq q$, and for integers q_r such that $1 \leq q_1 < q_2 < \cdots < q_k = q$, $q_0 = 0$, the matrices M_n, for $q_{r-1} < n \leq q_r$, all have the same order p_r then it is shown that the process may be reduced to multiplications and the inversion of k matrices of orders $p_1, \cdots, p_k$.

A code to solve (1.2) has been written by Max Goldstein for the I.B.M.-704 at New York University and this code has been successfully applied to a symmetric positive problem for the Tricomi equation.

This code has also been applied to solving pure elliptic problems to compare the direct method with iterative methods. The direct method was used, for instance to solve the bi-harmonic boundary problem of a simply supported rectangular slab. A comparison of running time would indicate that the direct method is considerably faster than iterative methods for this type of problem.

2. Direct methods. We seek a reduction of Q to the form

$$Q = LU, \tag{2.1}$$

where L and U are square matrices, partitioned in the same manner as Q, of the form

$$L = [C_n, I_n, 0]_1^q, \tag{2.2}$$

$$U = [0, A_n, E_n]_1^q, \tag{2.3}$$

where I_n is a unit matrix of the same order as M_n. We also partition the column vectors v and g in the same manner.

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_q \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_q \end{bmatrix}$$

Comparing right and left hand sides of (2.1) we set, for $1 < n \leq q$,

$$A_1 = M_1, \qquad D_n = C_n A_{n-1}, \qquad M_n = C_n E_{n-1} + A_n .$$

If the A_n are non-singular, the A_n and C_n may be obtained recursively from

$$A_n = M_n - D_n A_{n-1}^{-1} E_{n-1}, \qquad A_1 = M_1 \tag{2.4}$$

$$C_n = D_n A_{n-1}^{-1}, \qquad 1 < n \leq q. \tag{2.5}$$

To solve for v let $Uv = y$, then $Ly = g$; y and v may then be obtained recursively from

$$y_n = g_n - C_n y_{n-1} \qquad 1 < n \leq q, \tag{2.6}$$

where

$$y_1 = g_1, \quad \text{and}$$

$$v_n = A_n^{-1}(y_n - E_n v_{n+1}) \qquad 1 \leq n < q, \tag{2.7}$$

where

$$v_q = A_q^{-1} y_q .$$

We will refer to this recursive method (2.4)-(2.7) as an LU-process. It should be noted that this process requires the inversion of $A_1, A_2, \cdots, A_q$ and their storage for use on the "backward sweep" (2.7).

In certain cases most of these inversions may be avoided. For instance, if all the matrices M_n are of the same order p and the D_n are easily invertible, we may premultiply Q by the quasi-diagonal matrix

$$\begin{bmatrix} I & & & & 0 \\ & D_2^{-1} & & & \\ & & D_3^{-1} & & \\ & & & \ddots & \\ 0 & & & & D_q^{-1} \end{bmatrix}.$$

We may thus assume that $D_n = I$, $2 \leq n \leq q$. If we let $A_n = H_{n-1}^{-1} H_n$ then the LU-process becomes

$$H_n = H_{n-1} M_n - H_{n-2} E_{n-1} \qquad 2 \leq n \leq q, \tag{2.8}$$

where $H_0 = I$, $H_1 = M_1$;

$$H_{n-1}(y_n - g_n) = -H_{n-2} y_{n-1}, \qquad 2 \leq n \leq q, \tag{2.9}$$

where $y_1 = g_1$ and $H_q v_q = H_{q-1} y_q$.

If we let $u_n = H_{n-1} y_n$ then from (2.9)

$$u_n = H_{n-1} g_n - u_{n-1}, \qquad 1 < n \leq q, \tag{2.10}$$

where $u_1 = g_1$. To obtain the v_n we go back to the original equation (1.2) whence

$$v_{n-1} = g_n - M_n v_n - E_n v_{n+1}, \qquad 1 \leq n < q - 1 \tag{2.11}$$

where $v_{q-1} = g_q - M_q v_q$ and v_q is solved from

$$H_q v_q = u_q . \tag{2.12}$$

We have, in this case, only one matrix H_q of order p to invert.

This recursion (2.8)-(2.12) will be called an H-process. This method may be used, for instance, in solving problems for the Poisson or bi-harmonic equations over a rectangle. In the case of Poisson's equation the q inversions of the LU-process are reduced to solving only one set of p equations where p is the number of mesh points on a horizontal line. This fact was noted for these equations by Cornock [2] in specific examples.

For regions which are made up of rectangles a similar result may be obtained. For instance, in solving Poisson's equation for an L-shaped region made up of two rectangles R_1, R_2 each having p_1, p_2 points on a horizontal mesh line, respectively, only two matrices of order p_1 and p_2 need be inverted.

To treat the general case let us assume that for $1 \leq q_1 < q_2 < \cdots < q_k = q$, $q_0 = 0$ the M_n have the same order p_r for $q_{r-1} < n \leq q_r$. The matrix Q can then be partitioned again by combining those M_n having the same order. That is, let

$$Q = [D_n', M_n', E_n']_1^k,$$

where

$$D_{r+1}' = \begin{bmatrix} 0 & \cdots & 0 & D_{q_r+1} \\ & & & 0 \\ & & & \vdots \\ 0 & & & 0 \end{bmatrix}, \qquad 1 \le r < k,$$

$$M_{r+1}' = [D_n, M_n, E_n]_{q_r+1}^{q_{r+1}}, \qquad 0 \le r < k$$

$$E_{r+1}' = \begin{bmatrix} 0 & & & 0 \\ \vdots & & & \\ 0 & & & \\ E_{q_{r+1}} & 0 & \cdots & 0 \end{bmatrix}, \qquad 0 \le r < k - 1.$$

The LU-process can be performed for this new form for Q. Let g_r', y_r', v_r', $1 \le r \le k$ be corresponding column vectors of dimension p_r repartitioned as Q, where, for instance, we denote

$$v_{r+1}' = \begin{bmatrix} v_{q_r+1} \\ \vdots \\ v_{q_{r+1}} \end{bmatrix}, \qquad 0 \le r < k.$$

Let $y_1' = g_1'$, $y_2' = g_2' - C_2' y_1'$ where $D_2' = C_2' A_1'$. If we let

$$y_1' = A_1' w_1' \tag{2.13}$$

w_1' may be obtained by an H-process for (2.13). If we denote the H matrices that enter here by $H_1, \cdots, H_{q_1}$ then only H_{q_1} need be inverted and

$$y_2' = \begin{bmatrix} g_{q_1+1} - D_{q_1+1} w_{q_1} \\ g_{q_1+2} \\ \vdots \\ g_{q_2} \end{bmatrix}.$$

Thus it is seen that only the last vector component w_{q_1} of w_1' is needed and the reverse sweep given by (2.11) may be omitted. It will also appear later that only w_{q_1} and $H_{q_1}^{-1} H_{q_1-1}$ need be saved for future use. (It should be noted that where an H process is used the D matrices in M_n' are assumed to be non-singular and that this process is feasible only if these D's are easily, if not explicitly, invertible.)

To obtain y_3' we must compute A_2', which would generally require the inversion of A_1'. This however is not the case here. Since if we decompose A_1' by the method of (2.1), (2.2), (2.3) into $A_1' = L_1 U_1$ then a straightforward computation shows that

$$D_2'(A_1')^{-1} E_1' = D_2' U_1^{-1} L_1^{-1} E_1' = \begin{bmatrix} D_{q_1+1} H_{q_1}^{-1} H_{q_1-1} E_{q_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus A_2' differs from M_2' only in that M_{q_1+1} is replaced by

$$M_{q_1+1} - D_{q_1+1}H_{q_1}^{-1}H_{q_1-1}E_{q_1} .$$

That is A_2' is also of the form (1.1) and has all of its M matrices of the same order p_2 . We are therefore reduced to the previous case. We may proceed in this manner until all the y_r' , $1 \leq r \leq k$ have been computed and we note that $v_k' = w_k'$. In this last computation, of course, (2.11) will have to be used to get all of v_k' .

To obtain the other v_r' we note from (2.7) that, for $r < k$,

$$v_r' = w_r' - (A_r')^{-1}E_r'v_{r+1}' = w_r' - z_r' ,$$

where we let z_r' be the solution of

$$A_r'z_r' = E_r'v_{r+1}' . \tag{2.14}$$

If the last vector component v_{q_r} of v_r' would be known the other components would be obtainable from the original equations as in (2.11). Since

$$E_r'v_{r+1}' = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_{q_r}v_{q_r+1} \end{bmatrix}$$

if follows from (2.10) and (2.12) that if an H-process were carried out for (2.14) then

$$z_{q_r} = H_{q_r}^{-1}H_{q_r-1}E_{q_r}v_{q_r+1} ,$$

where $H_{q_r}^{-1}H_{q_r-1}$ is the matrix that entered in the computation of w_{q_r} . Thus if these are saved from before, the remaining v_r' are obtained without any further matrix inversions.

Assuming therefore that the work to invert the D-matrices in the M_n' is negligible we see that for a problem involving k sets of M-matrices of equal order only the k matrices $H_{q_1}, H_{q_2}, \cdots, H_{q_k}$ need be inverted. Thus in solving Poisson's or the bi-harmonic equation over a region made up of k adjoining rectangles the inversion of at most k relatively small matrices is required. A similar statement applies in higher dimensions.

We note that in the particular case where Q is the discrete Laplacian, where $M_n = J$, $n = 1, 2, \cdots, q$, D_n , E_n are identities and $J = [1, -4, 1]_1^p$, the inverse of H_q can be given explicitly [4]. If we denote the allied Chebyshev polynomials by

$$h_q(a) = \frac{\sinh (q + 1)x}{\sinh x}, \qquad 2 \cosh x = a,$$

then $H_q = h_q(J)$. The eigenvalues of J and H_q are given by

$$\lambda_m = 2 \cos \frac{m\pi}{p + 1} - 4, \qquad h_q(\lambda_m), \qquad m = 1, 2, \cdots, p,$$

respectively, and the matrix of normalized eigenvectors is

$$G = \left(\frac{2}{p + 1}\right)^{1/2} \left\{\sin \frac{km\pi}{p + 1}\right\}_{k,m=1}^{p} .$$

If we let

$$L = [h_q(\lambda_1) , \cdots , h_q(\lambda_p)]$$

be a diagonal matrix then

$$H_q^{-1} = GL^{-1}G.$$

For large problems the H_q may be ill-conditioned. For the above problem the P-condition $P(H_q)$, or ratio of largest to smallest eigenvalue of H_q is

$$P(H_q) \sim \frac{\pi \sinh 6 (q + 1)}{(p + 1) \sinh 6 \cdot \sinh \left(\frac{q + 1}{p + 1} \pi\right)},$$

so that this may get quite large for large $q = p$. The inversion of H_q may then present severe difficulties. It may however be feasible to break the problem into k groups as indicated above even though all the M_n have the same order.

Since (2.11) represents a marching process there is also the possibility of severe loss in accuracy in the value of v_1 for large q. The value of p does not appear to be an important factor in this loss for the discrete Laplacian problem. In the cases tried for $q = 5$, a 704 code yielded results accurate to at least five significant digits for values of $p = 5, 10, 20, 40$.

3. Criterion for decomposition. A sufficient condition for the validity of the decomposition is similar to that given for the LDU theorem [3]. Let

$$Q_1 = M_1 , \qquad Q_2 = \begin{pmatrix} M_1 E_1 \\ D_2 M_2 \end{pmatrix}, \cdots ,$$

$$Q_k = [D_n , M_n , E_n]_1^k , \cdots , Q_q = Q.$$

If $Q_1 , Q_2 , \cdots , Q_q$ *are non-singular then the decomposition* (2.1)-(2.3) *exists, and is uniquely given by the recursion* (2.4), (2.5). *In this case* $\det Q = \prod_{k=1}^{q} \det A_k$ *and if the* M_k *all have the same order and* $D_2 , \cdots , D_q$ *are non-singular then* $\det Q = \det H_q$.

The proof follows easily by induction. From the Schur-Frobenius formula

$$\det Q_{q+1} = \det \begin{bmatrix} Q_q & K \\ R & M_{q+1} \end{bmatrix}$$

$$= \det Q_q \cdot \det (M_{q+1} - RQ_q^{-1}K)$$

where $R = (0 \cdots 0 \ D_{q+1})$, $K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_q \end{bmatrix}$. From the inductive hypothesis

$$Q_q^{-1} = \begin{bmatrix} A_1^{-1} & & * \\ & \ddots & \\ 0 & & A_q^{-1} \end{bmatrix} \begin{bmatrix} I_1 & & 0 \\ & \ddots & \\ * & & I_q \end{bmatrix},$$

so that $RQ_q^{-1}K = D_{q+1}A_q^{-1}E_q$ and $\det Q_{q+1} = \det Q_q \cdot \det A_{q+1}$. This implies that A_{q+1} is non-singular and also proves the formulas for det Q.

A sufficient condition for $Q_1, \cdots, Q_q$ to be non-singular is, clearly, that for all u

$$\gamma u^T Q u \geq u^T u \tag{3.1}$$

for some non-zero constant γ.

4. Applications. We consider, as an application of the above processes, the problem of solving a certain boundary problem for the Tricomi equation[1]

$$Tu \equiv yu_{xx} - u_{yy} - f(x, y) = 0 \tag{4.1}$$

by a difference approximation given by Friedrichs [5]. These difference approximations for symmetric positive systems have been further investigated by C. K. Chu [1].

The problem for (4.1) is posed for the parallelogram

$$P: | y - x | \leq t, \qquad | x | \leq r; t, \qquad r > 0 \tag{4.2}$$

such that

$$Tu = 0, \qquad (x, y) \in P \tag{4.3}$$

$$u_x + u_y = 0 \quad \text{for} \quad | x - y | = t, \qquad | x | \leq r \tag{4.4}$$

$$u_y = 0 \quad \text{for} \quad x = -r, \qquad | y - x | \leq t. \tag{4.5}$$

No condition is specified on $x = r$.

Since the treatment given by Friedrichs calls for a rectangular region, P is transformed by $\xi = x$, $\eta = y - x$ into the rectangle

$$| \xi | \leq r, \qquad | \eta | \leq t \tag{4.6}$$

and the equation (4.1) is written as the system

$$\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} v_\xi - \begin{bmatrix} y & 1 \\ 1 & 1 \end{bmatrix} v_\eta = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{4.7}$$

where $y = \xi + \eta$, $v = (v_1, v_2)^T$, $v_1 = u_x$, $v_2 = u_y$. It is shown in [1] and [5] that after a premultiplication of (4.7) by a suitable two by two matrix of the form

$$\begin{bmatrix} \rho & y \\ 1 & \rho \end{bmatrix},$$

(4.7) and its boundary conditions can be brought into the required symmetric positive form. That is (4.1), (4.4), (4.5) can be written in the form

$$\tfrac{1}{2}(\alpha^\xi v_\xi + \alpha^\xi v_\eta + (\alpha^\xi v)_\xi + (\alpha^\xi v)_\eta) + \kappa v = g, \tag{4.8}$$

where for a constant ϵ

$$\alpha^\xi = \begin{bmatrix} \rho y & y \\ y & \rho \end{bmatrix}, \qquad \rho = 1 + \epsilon\eta, \tag{4.9}$$

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[1] A subscript of x, y, ξ or η indicates partial derivative.

$$\alpha^\eta = -\begin{bmatrix} (1+\rho)y & \rho+y \\ \rho+y & 1+\rho \end{bmatrix}, \tag{4.10}$$

$$g = \begin{bmatrix} \rho f \\ f \end{bmatrix}, \tag{4.11}$$

$$\kappa = -\tfrac{1}{2}(\alpha^\xi_\xi + \alpha^\eta_\eta) = \tfrac{1}{2}\begin{bmatrix} 1+\epsilon y & \epsilon \\ \epsilon & \epsilon \end{bmatrix}. \tag{4.12}$$

As is shown in [1] and [5] the boundary conditions can be written in the form

$$\beta v = \mu v, \tag{4.13}$$

where

$$\beta = -\alpha^\xi, \qquad \mu = -\alpha^\xi_- = -\begin{bmatrix} \rho y & y \\ y & \dfrac{2y}{\rho} - \rho \end{bmatrix} \quad \text{for} \quad \xi = -r, \tag{4.14}$$

$$\beta = \alpha^\xi, \qquad \mu = \alpha^\xi_+ = \alpha^\xi \quad \text{for} \quad \xi = r \tag{4.15}$$

$$\beta = -c^\eta, \qquad \mu = -\alpha^\eta_- \quad \text{for} \quad \eta = -t \quad \text{where} \tag{4.16}$$

$$\alpha^\eta_- = \frac{1}{\rho-1}\begin{bmatrix} 2\rho^2 - y(\rho^2+1) & (1+\rho)(\rho-y) \\ (1+\rho)(\rho-y) & \rho^2 - 2y + 1 \end{bmatrix}$$

$$\beta = \alpha^\eta, \qquad \mu = \alpha^\eta_- \quad \text{for} \quad \eta = t. \tag{4.17}$$

To obtain the difference equations we divide the intervals $(-r, r)$, $(-t, t)$ by the $2p - 1$, $2q - 1$ points,

$$\xi_i = (i - p)\,\Delta\xi, \qquad 1 \le i \le 2p - 1, \qquad 1 < p, \tag{4.18}$$

$$\eta_j = (j - q)\,\Delta\eta, \qquad 1 \le j \le 2q - 1, \qquad 1 < q, \tag{4.19}$$

respectively, where p and q are odd integers and

$$\Delta\xi = \frac{r}{p-1}, \qquad \Delta\eta = \frac{t}{q-1}$$

are the respective interval lengths.

According to [5] an equation is written only for the pq odd numbered points (ξ_i, η_j), $i = 2m - 1$, $j = 2n - 1$, $1 \le m \le p$, $1 \le n \le q$. If we write $v_{ij} = v(\xi_i, \eta_j)$, $\alpha^\xi_{ij} = \alpha^\xi(\xi_i, \eta_j)$ and similarly for the other variables then the difference equations are given for an interior point $1 < m < p$, $1 < n < q$, by

$$\frac{1}{2h}(\alpha^\xi_{i+1,j}v_{i+2,j} - \alpha^\xi_{i-1,j}v_{i-2,j}) + \frac{1}{2k}(\alpha^\xi_{i,j+1}v_{i,j+2} - \alpha^\xi_{i,j-1}v_{i,j-2}) + \kappa_{ij}v_{ij} = g_{ij}, \tag{4.20}$$

where $h = 2\,\Delta\xi$, $k = 2\,\Delta\eta$.

For a boundary point at least one of the subscripts $i \pm 1$, $j \pm 1$ falls outside the range prescribed. In that case the α and v of the corresponding term are both evaluated at that boundary point and then replaced according to the rule (4.13). The $2h$ and/or the $2k$ in the difference involved are also replaced by h and/or k, respectively.

As an illustration we consider the equation for the point (ξ_1, η_j), not a corner:

$$\frac{1}{h}(\alpha^{\xi}_{2,j}v_{3,j} - (\alpha^{\xi}_{-})_{1j}v_{1j}) + \frac{1}{2k}(\alpha^{\eta}_{1,j+1}v_{1,j+2} - \alpha^{\eta}_{1,j-1}v_{1,j-2}) + \kappa_{1j}v_{1j} = g_{1j}\,. \tag{4.21}$$

For the corner point (ξ_1, η_1), j, $2k$, $\alpha^{\eta}_{1,j-1}v_{1,j-2}$ are replaced by 1, k and $(\alpha^{\eta}_{-})_{11}v_{11}$, respectively in (4.21). The other types of boundary points are treated in a similar manner.

Since each (m, n) yields one equation we obtain pq equations in the pq unknowns v_{ij}, or $2pq$ scalar equations for $2pq$ scalar unknowns. For each of the equations we note that v_{ij} appears with at most four of its neighbors $v_{i+2,i}$, $v_{i-2,j}$, $v_{i,j-2}$ $v_{i,j+2}$. If we set $v^n_m = v_{ij}$ and denote by v^n the vector with $2p$ scalar components arising from the points (ξ_i, η_{2n-1}) on a horizontal line, the difference equations take the form

$$d^n_m v^{n-1}_m + a^n_m v^n_{m-1} + b^n_m v^n_m + c^n_m v^n_{m+1} + e^n_m v^{n+1}_m = g^n_m \tag{4.22}$$

$$1 \le m \le p, \qquad 1 \le n \le q.$$

This system can be written in the form (1.1), (1.2) where

$$M_n = [a^n_m, b^n_m, c^n_m]^p_{m=1},$$

$$D_n = [0, d^n_m, 0]^p_{m=1},$$

$$E_n = [0, e^n_m, 0]^p_{m=1},$$

$$v = \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^q \end{bmatrix}, \qquad g = \begin{bmatrix} g^1 \\ g^2 \\ \vdots \\ g^q \end{bmatrix},$$

and where for $1 < m < p$, $1 < n < q$

$$d^n_m = -\frac{1}{2k}\alpha^{\eta}_{i,j-1}, \qquad a^n_m = -\frac{1}{2h}\alpha^{\xi}_{i-1,j}$$

$$b^n_m = \kappa_{ij}, \qquad c^n_m = \frac{1}{2h}\alpha^{\xi}_{i+1,j}$$

$$e^n_m = \frac{1}{2k}\alpha^{\eta}_{i,j+1}, \qquad g^n_m = g_{ij}\,.$$

For the boundary points $n = 1, q$, $1 \le m \le p$,

$$d^1_m = 0, \qquad d^q_m = -\frac{1}{k}\alpha^{\eta}_{i,2q-2}$$

$$e^1_m = \frac{1}{k}\alpha^{\eta}_{i,2}, \qquad e^f_m = 0$$

and for $m = 1, p$, $1 \le n \le q$

$$a^n_1 = 0, \qquad a^n_p = -\frac{1}{h}\alpha^{\xi}_{2p-2,j}$$

$$c^n_1 = \frac{1}{h}\alpha^{\xi}_{2,j}, \qquad c^n_p = 0.$$

For $m = 1, p, 1 < n < q$,

$$b_1^n = \kappa_{ij} - \frac{1}{h}(\alpha_-^\xi)_{ij}, \qquad b_p^n = \kappa_{ij} + \frac{1}{h}\alpha_{ij}^\xi$$

and for corner points

$$b_m^n = \kappa_{ij} - \frac{1}{h}(\alpha_-^\xi)_{ij} - \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = 1, \qquad n = 1,$$

$$b_m^n = \kappa_{ij} + \frac{1}{h}\alpha_{ij}^\xi - \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = p, \qquad n = 1,$$

$$b_m^n = \kappa_{ij} - \frac{1}{h}(\alpha_-^\xi)_{ij} + \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = 1, \qquad n = q,$$

$$b_m^n = \kappa_{ij} + \frac{1}{h}\alpha_{ij}^\xi + \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = p, \qquad n = q,$$

where in all cases $i = 2m - 1$, $j = 2n - 1$.

Thus this boundary problem for the Tricomi equation yields a matrix of the form (1.1). It has been shown by Chu [1] that, if ϵ, r, t are properly chosen, e.g., $\epsilon = 1/2$, $r = 1/2$, $t = 1/5$, this matrix satisfies the inequality (3.1). The LU-process is therefore applicable. Since the M_n all have the same order we may in fact use the H-process providing the D_n are non-singular and easily invertible. The D_n are quasi-diagonal, and a simple computation shows that for the above choice of ϵ, r, t, α^η is non-singular.

The above mentioned code was used to solve the problem described above for the choice of

$$f(x, y) = 6yx - 4y^2 + y - 1 - 2x$$

and was run for various values of p and q. The solution to the analytic problem (4.3)-(4.5) can be given explicitly by

$$u(x, y) = (y - x)^2(\tfrac{1}{2} + x) - x/25 \tag{4.23}$$

and for (4.8)-(4.17) it is given by

$$\begin{aligned} v_1 &= \eta^2 - \eta(1 + 2\xi) - 1/25, \\ v_2 &= \eta(1 + 2\xi). \end{aligned} \tag{4.24}$$

For the case of $p = 15$, $q = 15$ the code yielded an answer accurate generally to three significant digits. The values given at $I(0, 0)$, $II(1/2, 0)$ by the code are illustrated by Tables I, II respectively:

$p \times q$		3×3	5×5	7×13	11×7	15×15	
I	v_1	-6.22	-4.07	-4.02	-4.27	-4.04	$\times 10^{-2}$
	v_2	-0.37	0.08	0.01	-0.04	-0.01	
II	v_1	-7.13	-4.96	-4.34	-3.99	-3.99	$\times 10^{-2}$
	v_2	-1.01	0.54	0.14	-0.21	-0.05	

The exact solution given by (4.24) for $\eta = 0$ is $v_1 = -.04$, $v_2 = 0$.

Other problems for the Tricomi equation were run to check the influence of the positivity and boundary conditions. In one case the two by two premultiplier matrix, needed to guarantee "positivity," was omitted. Again an approximate solution was obtained with a slight loss in accuracy. For instance the value at I given by the code was $v_1 = -.0454$, $v_2 = -.0019$, for $p = 11$, $q = 7$.

A problem for the homogeneous Tricomi equation with inhomogeneous boundary conditions $(\mu - \beta)v = f$, for a given f, also yielded results similar to those given above.

Symmetric positive systems for dimensions higher than two may be treated in a similar manner.

As is pointed out in [2] and [4], problems for the bi-harmonic equation can also be put into the form (1.1). The LU-method was carried out for a simply supported rectangular plate and was found to give results accurate to about three significant digits for a 30 by 30 mesh. The running time for this problem was about an hour on the IBM-704.

The above methods may also be combined with iterative or group relaxation methods where each individual group relaxation is done by a direct method. This has already been proposed for a multigroup diffusion problem by Nohel and Timlake [6].

It may also be noted that, in problems where higher order difference schemes are available, the direct methods will require relatively little additional operations and thus one may require fewer mesh points in a given problem.

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