# A GENERAL THEOREM CONCERNING THE STABILITY OF A PARTICULAR NON-NEWTONIAN FLUID*

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**Summary.** It is the intention of the present paper to prove a theorem concerning the stability of a particular non-Newtonian fluid suggested to the author by Professor R. S. Rivlin of Brown University. The method used in proving this theorem is similar to that employed by H. Schlichting in his proof of a similar theorem for an inviscid fluid which was originally established by Lord Rayleigh. The acceleration gradients introduced by the non-Newtonian fluid model into the constitutive equations are found to alter the stability criterion set forth by Rayleigh for an inviscid fluid.

**1. Introduction.** As early as 1880 Rayleigh [1] proved that for an inviscid fluid the existence of a point of inflection in the velocity profile of a steady one-dimensional basic flow is a necessary condition for the growth of a superimposed two-dimensional disturbance. It is the intention of the present paper to prove a similar theorem for a particular non-Newtonian fluid suggested to the author by Professor R. S. Rivlin.* The method used in proving this theorem is similar to that employed by Schlichting [2] in his proof of the Rayleigh theorem.

**2. The constitutive equations and equations of motion.** Let $X_j(j = 1, 2, 3)$ be the coördinates referred to a rectangular Cartesian coordinate system $x_i$, of a generic particle of a continuous medium in the undeformed state at time $t_0$. Let $x_i$ be the coordinates of the same particle in the deformed state at time $t$. It then follows that the components of velocity $v_i$ and the components of acceleration $a_i$ of the particle are given by

$$v_i = \frac{\partial x_i}{\partial t} \quad \text{and} \quad a_i = \frac{\partial^2 x_i}{\partial t^2}, \tag{2.1}$$

where $x_i$ are regarded as single-value continuous functions of $X_k(k = 1, 2, 3)$ and $t$, having as many continuous derivatives as the analysis requires. As is well known, if $v_i$ are considered to be functions of $x_k$ and $t$, then

$$a_i = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}. \tag{2.2}$$

Now the equations of motion are

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) + \rho f_i = \frac{\partial t_{il}}{\partial x_l}, \qquad (l = 1, 2, 3), \tag{2.3}$$

where $\rho$ is the mass of the medium per unit volume and $f_i$ are the components in the coordinate directions of the body force per unit mass, also measured in the deformed state. The components of stress $t_{il}$ resulting from the deformation are defined as follows; $t_{i1}$, $t_{i2}$ and $t_{i3}$ are the components of the force per unit area in the positive direction of the $x_1$, $x_2$ and $x_3$ axes respectively, measured in the deformed state, exerted across an

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element of area at $(x_1, x_2, x_3)$ normal to the $x_j$ axis, by the material on the positive side of the element upon the material on the negative side of the element.

Rivlin [3] showed that if $t_{jl}$ at the point $x_k$ and at time $t$ are assumed to be polynomials in the velocity gradients $\partial v_m/\partial x_n$ $(m, n = 1, 2, 3)$ and the acceleration gradients $\partial a_m/\partial x_n$ and if, in addition, the medium is assumed to be isotropic at time $t_0$, then the stress matrix $\mathbf{T} = \| t_{jl} \|$ is expressible in the form

$$\begin{aligned} \mathbf{T} = \varphi_0\mathbf{I} + \varphi_1\mathbf{A}_1 + \varphi_2\mathbf{A}_2 + \varphi_3\mathbf{A}_1^2 + \varphi_4\mathbf{A}_2^2 + \varphi_5(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) \\ + \varphi_6(\mathbf{A}_1^2\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1^2) + \varphi_7(\mathbf{A}_1\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1) + \varphi_8(\mathbf{A}_1^2\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1^2), \end{aligned} \tag{2.4}$$

where $\mathbf{I}$ is the unit matrix, $\mathbf{A}_1$ and $\mathbf{A}_2$ are symmetric kinematic matrices defined by

$$\mathbf{A}_1 = \left\| \frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j} \right\| \quad \text{and} \quad \mathbf{A}_2 = \left\| \frac{\partial a_j}{\partial x_l} + \frac{\partial a_l}{\partial x_j} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial v_m}{\partial x_l} \right\| \tag{2.5}$$

and $\varphi_q$ $(q = 0, 1, 2, \cdots, 8)$ are polynomials in $tr\mathbf{A}_1$, $tr\mathbf{A}_2$, $tr\mathbf{A}_1^2$, $tr\mathbf{A}_2^2$, $tr\mathbf{A}_1^3$, $tr\mathbf{A}_2^3$, $tr\mathbf{A}_1\mathbf{A}_2$, $tr\mathbf{A}_1^2\mathbf{A}_2$, $tr\mathbf{A}_1\mathbf{A}_2^2$ and $tr\mathbf{A}_1^2\mathbf{A}_2^2$. In a later paper Rivlin [4] pointed out that for an incompressible material, the stress corresponding to a specific state of flow is indeterminate to the extent of an arbitrary hydrostatic pressure $p$. Since in the present paper we shall confine our analysis to an incompressible fluid, we may replace $\varphi_0$ in equation (2.4) by $-p$. We shall also restrict our investigation to a fluid for which $\varphi_1$ and $\varphi_2$ are constants and $\varphi_q$ $(q = 3, 4, 5, \cdots, 8)$ are identically zero.

Equation (2.4) now takes the form

$$\mathbf{T} = -p\mathbf{I} + \varphi_1\mathbf{A}_1 + \varphi_2\mathbf{A}_2, \tag{2.6}$$

or alternatively

$$t_{jl} = -p\,\delta_{jl} + \varphi_1\left(\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j}\right) + \varphi_2\left(\frac{\partial a_j}{\partial x_l} + \frac{\partial a_l}{\partial x_j} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial v_m}{\partial x_l}\right). \tag{2.7}$$

Introducing Eqs. (2.2) into Eqs. (2.7), we obtain

$$\begin{aligned} t_{jl} = -p\,\delta_{jl} + \varphi_1\left(\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j}\right) + \varphi_2\left(\frac{\partial^2 v_j}{\partial t\,\partial x_l} + \frac{\partial v_m}{\partial x_l}\frac{\partial v_j}{\partial x_m}\right. \\ \left. + v_m\frac{\partial^2 v_j}{\partial x_l\,\partial x_m} + \frac{\partial^2 v_l}{\partial t\,\partial x_j} + \frac{\partial v_m}{\partial x_j}\frac{\partial v_l}{\partial x_m} + v_m\frac{\partial^2 v_l}{\partial x_j\,\partial x_m} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial v_m}{\partial x_l}\right). \end{aligned} \tag{2.8}$$

Since we have assumed that the fluid is incompressible, the continuity equation is

$$\frac{\partial v_m}{\partial x_m} = 0. \tag{2.9}$$

Introducing the constitutive equations (2.8) in the equations of motion (2.3), employing the incompressibility condition (2.9) and neglecting body forces, we obtain

$$\begin{aligned} \rho\left(\frac{\partial v_j}{\partial t} + v_k\frac{\partial v_j}{\partial x_k}\right) = -\frac{\partial p}{\partial x_j} + \varphi_1\frac{\partial^2 v_j}{\partial x_l\,\partial x_l} + \varphi_2\left(\frac{\partial^3 v_j}{\partial t\,\partial x_l\,\partial x_l}\right. \\ + \frac{\partial v_j}{\partial x_m}\frac{\partial^2 v_m}{\partial x_l\,\partial x_l} + 2\frac{\partial v_m}{\partial x_l}\frac{\partial^2 v_j}{\partial x_l\,\partial x_m} \\ + v_m\frac{\partial^3 v_j}{\partial x_l\,\partial x_l\,\partial x_m} + 2\frac{\partial v_m}{\partial x_l}\frac{\partial^2 v_l}{\partial x_l\,\partial x_j\,\partial x_m} \\ \left. + 2\frac{\partial v_m}{\partial x_l}\frac{\partial^2 v_m}{\partial x_l\,\partial x_j\,\partial x_l} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial^2 v_m}{\partial x_l\,\partial x_l\,\partial x_l}\right). \end{aligned} \tag{2.10}$$

**3. Development of the stability equation.** Consider next a two-dimensional steady laminar flow with velocity components

$$W_1 = W_1(x_2) \quad \text{and} \quad W_2 = W_3 = 0. \tag{3.1}$$

Examples of such flows are the flow between a pair of parallel plates sufficiently removed from the intake section, and the flow in the boundary layer along a flat plate excluding the region of its leading edge. The pressure $P$ necessary to maintain such flows is in general a function of both $x_1$ and $x_2$.

We shall superimpose upon the laminar flow a small two dimensional disturbance with velocity components and associated pressure given by

$$u_1 = u_1(x_1, x_2, t), \qquad u_2 = u_2(x_1, x_2, t), \qquad u_3 = 0 \quad \text{and} \quad p^* = p^*(x_1, x_2, t). \tag{3.2}$$

Thus the velocity components and pressure of the composite flow are

$$v_1 = W_1 + u_1, \qquad v_2 = u_2, \qquad v_3 = 0 \quad \text{and} \quad p = P + p^*. \tag{3.3}$$

Further, we shall require that the composite flow satisfy the same boundary conditions as the steady laminar flow. Thus the disturbance satisfies the boundary conditions

$$u_1 = u_2 = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = L \tag{3.4}$$

in the case of flow between parallel plates separated by a distance $L$, and

$$u_1 = u_2 = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 \to \infty \tag{3.5}$$

in the case of boundary layer flow along a flat plate.

Next, introducing the velocity components and pressure given by (3.3) into Eqs. (2.9) and (2.10), and assuming that the velocity components $u_i$ are sufficiently small so that terms of the second degree in $u_i$ and derivatives of $u_i$ may be neglected in comparison with terms of the first degree, we obtain

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{3.6}$$

$$\begin{aligned} \rho\left(\frac{\partial u_1}{\partial t} + W_1 \frac{\partial u_1}{\partial x_1} + W_1' u_2\right) = {} & -\frac{\partial P}{\partial x_1} - \frac{\partial p^*}{\partial x_1} + \varphi_1(\nabla^2 u_1 + W_1'') \\ & + \varphi_2\left(\nabla^2 \frac{\partial u_1}{\partial t} + 2W_1'' \frac{\partial u_2}{\partial x_2} + 4W_1' \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right. \\ & + W_1' \nabla^2 u_2 + 3W_1'' \frac{\partial u_1}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_1}{\partial x_1} \\ & \left. + W_1''' u_2 + 2W_1' \frac{\partial^2 u_2}{\partial x_1^2}\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \rho\left(\frac{\partial u_2}{\partial t} + W_1 \frac{\partial u_2}{\partial x_1}\right) = {} & -\frac{\partial P}{\partial x_2} - \frac{\partial p^*}{\partial x_2} + \varphi_1 \nabla^2 u_2 + \varphi_2\left(\nabla^2 \frac{\partial u_2}{\partial t}\right. \\ & + 4W_1' \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + 3W_1'' \frac{\partial u_2}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_2}{\partial x_1} \\ & \left. + 4W_1' W_1'' + 4W_1'' \frac{\partial u_1}{\partial x_2} + 2W_1' \frac{\partial^2 u_1}{\partial x_2^2} + 2W_1' \nabla^2 u_1\right), \end{aligned} \tag{3.8}$$

where $\nabla^2 = (\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2)$, and primes denote ordinary differentiation with respect to $x_2$.

Further, introducing Eq. (3.6) into Eqs. (3.7) and (3.8) and assuming the basic laminar flow satisfies the equations of motion, we obtain

$$\rho\left(\frac{\partial u_1}{\partial t} + W_1 \frac{\partial u_1}{\partial x_1} + W_1' u_2\right) = -\frac{\partial p^*}{\partial x_1} + \varphi_1 \nabla^2 u_1 + \varphi_2\left(\nabla^2 \frac{\partial u_1}{\partial t} + 3W_1' \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + W_1'' \frac{\partial u_1}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_1}{\partial x_1} + W_1''' u_2 + 3W_1' \frac{\partial^2 u_2}{\partial x_1^2}\right) \tag{3.9}$$

and

$$\rho\left(\frac{\partial u_2}{\partial t} + W_1 \frac{\partial u_2}{\partial x_1}\right) = -\frac{\partial p^*}{\partial x_2} + \varphi_1 \nabla^2 u_2 + \varphi_2\left(\nabla^2 \frac{\partial u_2}{\partial t} + 2W_1' \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + 3W_1'' \frac{\partial u_2}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_2}{\partial x_1} + 4W_1'' \frac{\partial u_1}{\partial x_2} + 4W_1' \frac{\partial^2 u_1}{\partial x_2^2}\right). \tag{3.10}$$

Differentiating Eq. (3.9) with respect to $x_2$ and Eq. (3.10) with respect to $x_1$, eliminating $\partial^2 p^*/\partial x_1 \partial x_2$ from the resulting equations and again employing Eq. (3.6), we arrive at

$$\left(\frac{\partial}{\partial t} + W_1 \frac{\partial}{\partial x_1} - \nu \nabla^2 - \gamma \nabla^2 \frac{\partial}{\partial t} - \gamma W_1 \nabla^2 \frac{\partial}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = (\gamma W'''' - W'')u_2 , \tag{3.11}$$

where $\nu = \varphi_1/\rho$ and $\gamma = \varphi_2/\rho$.

Further, we shall assume that the velocity components of the disturbance are of the form

$$u_i(x_1 , x_2 , t) = u_j^*(x_2) \exp [iA(x_1 - Ct)], \qquad (j = 1, 2) \tag{3.12}$$

where $| u_j^* u_j^* |$ is the amplitude and $A$ the wave number of the disturbance, and $C$ is a complex number. The real part of $C$, $C_r$, is the phase velocity of the disturbance and the imaginary part, $C_i$, is the amplification factor. If $C_i > 0$, the disturbance tends to grow; if $C_i < 0$, the disturbance decays; and if $C_i = 0$, the disturbance is neutral.

Introducing the velocity components (3.12) into Eqs. (3.6) and (3.11), we obtain

$$iAu_1^* + u_2^{*\prime} = 0 \tag{3.13}$$

and

$$iA(W_1 - C)(u_1^{*\prime} - iAu_2^*) - [\nu + iA\gamma(W_1 - C)](u_1^{*\prime\prime\prime} - A^2 u_1^{*\prime} - iAu_2^{*\prime\prime} + iA^3 u_2^*) = (\gamma W_1'''' - W_1'')u_2^* . \tag{3.14}$$

Replacing $u_1^*$ in Eq. (3.14) by $iu_2^{*\prime}/A$ from Eq. (3.13), we obtain

$$iA(W_1 - C)(u_2^{*\prime\prime} - A^2 u_2^*) - [\nu + iA\gamma(W_1 - C)](u_2^{*\prime\prime\prime\prime} - 2A^2 u_2^{*\prime\prime} + A^4 u_2^*) = iA(W_1'' - \gamma W_1'''')u_2^* . \tag{3.15}$$

Writing Eq. (3.15) in dimensionless form by letting $\eta_2 = x_2/L$, $V_1 = W_1/W_0$, $w_2 = u_2^*/W_0$, $\alpha = AL$ and $c = C/W_0$ where $L$ and $W_0$ are a characteristic length and a characteristic velocity respectively of the steady laminar flow, we arrive at

$$i\alpha(V_1 - c)(w_2'' - \alpha^2 w_2) - \left[\frac{1}{R} + \frac{i\alpha}{S}(V_1 - c)\right](w_2'''' - 2\alpha^2 w_2'' + \alpha^4 w_2) = i\alpha\left(V_1'' - \frac{1}{S}V_1''''\right)w_2 , \tag{3.16}$$

where $R = W_0 L/\nu$ is the Reynolds' number of the laminar flow, $S = L^2/\gamma$ is a non-dimensional parameter arising from the presence of non-Newtonian terms in the constitutive equations of the fluid, and primes denote ordinary differentiation with respect to $\eta_2$ .

We see that if $S \to \infty$, Eq. (3.16) becomes the familiar Orr-Sommerfeld stability equation.

Finally, in terms of $w_2$ the boundary conditions (3.4) and (3.5) become

$$w_2 = w_2' = 0 \quad \text{at} \quad \eta_2 = 0 \quad \text{and} \quad \eta_2 = 1, \tag{3.17}$$

and

$$w_2 = w_2' = 0 \quad \text{at} \quad \eta_2 = 0 \quad \text{and} \quad \eta_2 \to \infty \tag{3.18}$$

respectively.

**4. The general theorem.** Under the assumption that $S$ is finite and $R$ is infinite, Eq. (3.16) takes the form

$$(V_1 - c)(w_2'' - \alpha^2 w_2) - \frac{1}{S}(V_1 - c)(w_2'''' - 2\alpha^2 w_2'' + \alpha^4 w_2) = (V_1'' - \frac{1}{S}V_1'''')w_2 . \tag{4.1}$$

We shall now prove the following theorem.

*The existence of a point in the flow field for which $V_1'' - (1/S)V_1''''$ is equal to zero, is a necessary condition for the amplification of a disturbance.*

Regarding $w_2$ as a complex variable, we define

$$M(w_2) = \frac{1}{S}w_2'''' - \left(1 + \frac{2\alpha^2}{S}\right)w_2'' + \left(\alpha^2 + \frac{\alpha^4}{S}\right)w_2 + \frac{\left(V_1'' - \frac{1}{S}V_1''''\right)}{V_1 - c}w_2 \tag{4.2}$$

and

$$\bar{M}(w_2) = \frac{1}{S}\bar{w}_2'''' - \left(1 + \frac{2\alpha^2}{S}\right)\bar{w}_2'' + \left(\alpha^2 + \frac{\alpha^4}{S}\right)\bar{w}_2 + \frac{\left(V_1'' - \frac{1}{S}V_1''''\right)}{V_1 - \bar{c}}\bar{w}_2 , \tag{4.3}$$

where a bar denotes the complex conjugate of the corresponding unbarred quantity. It is easily seen from Eq. (4.1) that both $M(w_2)$ and $\bar{M}(w_2)$ are equal to zero.

To prove the theorem we assume that $V_1'' - (1/S)V_1'''' \neq 0$ throughout the flow field. Then, multiplying $M(w_2)$ by $\bar{w}_2$ and $\bar{M}(w_2)$ by $w_2$ and subtracting the resulting expressions, we obtain

$$\begin{aligned}\bar{w}_2 M(w_2) - w_2\bar{M}(w_2) = {} & \frac{1}{S}(\bar{w}_2 w_2'''' - w_2\bar{w}_2'''') \\ & - \left(1 + \frac{2\alpha^2}{S}\right)(\bar{w}_2 w_2'' - w_2\bar{w}_2'') \\ & + \left(V_1'' - \frac{1}{S}V_1''''\right)\left(\frac{1}{V_1 - c} - \frac{1}{V_1 - \bar{c}}\right) \mid w_2 \mid^2.\end{aligned} \tag{4.4}$$

Integrating Eq. (4.4) with respect to $\eta_2$ between the limits $\eta_2 = 0$ and $\eta_2 = 1$, we have

$$\int_0^1 [\bar{w}_2 M(w_2) - w_2 \bar{M}(w_2)]\, d\eta_2 = \frac{1}{S} [(\bar{w}_2 w_2''' - w_2 \bar{w}_2''') - (\bar{w}_2' w_2'' - w_2' \bar{w}_2'')] \Big|_0^1 - \left(1 + \frac{2\alpha^2}{S}\right)[\bar{w}_2 w_2' - w_2 \bar{w}_2'] \Big|_0^1 + 2ic_i \int_0^1 \frac{\left(V_1'' - \frac{1}{S} V_1''''\right)}{|V_1 - c|^2} |w_2|^2\, d\eta_2 . \tag{4.5}$$

Because of the boundary conditions (3.17), the first two terms on the right hand side of Eq. (4.5) vanish, and further, since both $M(w_2)$ and $\bar{M}(w_2)$ are equal to zero, the left hand side of the equation is equal to zero. It then follows that

$$c_i \int_0^1 \frac{\left(V_1'' - \frac{1}{S} V_1''''\right)}{|V_1 - c|^2} |w_2|^2\, d\eta_2$$

must vanish. But this is impossible, since $c_i > 0$, for a disturbance which tends to grow, and hence $[1/|V_1 - c|^2] > 0$. Further $|w_2|^2$ is positive and by assumption $V_1'' - (1/S)V_1'''' \neq 0$ everywhere in the flow field. Thus we conclude that for a disturbance which tends to grow, there exists an $\eta_2$, $0 < \eta_2 < 1$, for which $V_1'' - (1/S)V_1''''$ is equal to zero.

In the case of laminar flow by a flat plate the proof of the theorem differ only in that the limits of integration become $\eta_2 = 0$ and $\eta_2 = \infty$, and the boundary conditions (3.18) are used in place of those given by (3.17).

## References

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