FURTHER PROPERTIES OF CERTAIN CLASSES OF TRANSFER FUNCTIONS*

BY

ARMEN H. ZEMANIAN

College of Engineering, New York University

Abstract. Some previously defined classes of rational transfer functions are extended to transfer functions that need not be rational. Several additional properties of these types of functions are then developed. Finally, bounds on certain derivatives of the unit impulse responses corresponding to such functions are shown to exist.

1. Let $Z(s)$ denote the transfer function of a system and let $W(t)$ be the response of the system to a unit impulse applied at time $t = 0$. Then $Z(s)$ and $W(t)$ are related by the Laplace transform.

$$Z(s) = \int_0^\infty W(t) \exp(-st)\, dt. \tag{1}$$

It will be assumed henceforth that $W(t)$ and its derivatives are integrable in any finite interval. The transfer function $Z(s)$ is a function of the complex variable, $s = \sigma + j\omega$. Since $W(t)$ is a real function for physical systems, the complex singularities of $Z(s)$ occur in complex conjugate pairs.

In a recent series of papers [1, 2, 3] a number of properties were established for certain classes of rational transfer functions. One of the objects of this paper is to remove the restrictions that $Z(s)$ be rational. This will require a more general form for the definitions of the classes of transfer functions. Let $Z(s)$ be analytic for $\sigma \geq 0$ and let it be asymptotic to K/s^k as s approaches infinity where K is a positive constant and k is a positive integer. In the special case where $Z(s)$ is rational, we have

$$Z(s) = K \frac{s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} \tag{2}$$

$$k = m - n > 0. \tag{3}$$

As noted above, $W(t)$ assumes only real values. Consequently, all the coefficients in (2) must be real.

Returning to the general case where $Z(s)$ need not be rational, let $Z_q(s)$ denote the following successive integrations of $Z(s)$.

$$Z_q(s) = \int_s^\infty ds_{q-1} \int_{s_{q-1}}^\infty ds_{q-2} \cdots \int_{s_1}^\infty Z(s_0)\, ds_0 . \tag{4}$$

Only the principal branch of this multivalued function will be needed. In fact, it will be sufficient to assume that the arbitrary paths of integration in (4) never enter into the region defined by $\sigma \leq \sigma_1 < 0$ where σ_1 is the largest real part among the singularities of $Z(s)$. Under this restriction, (4) can be considered a single-valued function.

*Received June 19, 1959.

Also, let the real and imaginary parts of some subsequently needed quantities be denoted by

$$s = \sigma + j\omega, \tag{5}$$

$$s_q = \sigma_q + j\omega_q , \tag{6}$$

$$Z(j\omega) = R(\omega) + jI(\omega), \tag{7}$$

$$Z_q(j\omega) = R_q(\omega) + jI_q(\omega). \tag{8}$$

The following relations may then be obtained from (4). For q odd,

$$R_q(\omega) = (-1)^{(q+1)/2} \int_\omega^\infty d\omega_{q-1} \int_{\omega_{q-1}}^\infty d\omega_{q-2} \cdots \int_{\omega_1}^\infty I(\omega_0)\, d\omega_0 \tag{9}$$

and, for q even,

$$R_q(\omega) = (-1)^{q/2} \int_\omega^\infty d\omega_{q-1} \int_{\omega_{q-1}}^\infty d\omega_{q-2} \cdots \int_{\omega_1}^\infty R(\omega_0)\, d\omega_0 . \tag{10}$$

$R_q(\omega)$ and $I_q(\omega)$ are even and odd functions of ω, respectively [2; lemma 2].

We may now state the following definition.

Definition 1. *A function $Z(s)$ will be called a class k function if the following conditions hold.*

(a) *$Z(s)$ is analytic for $\sigma \geq 0$.*
(b) *$Z(s) \sim K/s^k$ as $s \to \infty$ where K is a positive constant and k is a positive integer.*
(c) *$Z_{k-1}(s)$ is a positive real function in the half plane, $\sigma \geq 0$.*

A function is said to be positive real if its real part is nonnegative for $\sigma \geq 0$ and if it assumes only real values along the real axis [4].

A certain subclass of each class k will be required in the following discussion. The definition of this subclass is again a generalization of the previously given one [2] in that the condition that $Z(s)$ be rational is dropped. Furthermore, $R(\omega)$ and $dI/d\omega$ are now allowed to assume the value of zero at $\omega = 0$.

Definition 2. *A function $Z(s)$ will be called a subclass k function if conditions* (a) *and* (b) *in Definition* 1 *hold and if the following condition holds.*

(c′) *For k odd, $R(\omega)$ has $k - 1$ changes of sign for $-\infty < \omega < \infty$ and $R(\omega)$ is positive in the neighborhood of $\omega = 0$; for k even, $I(\omega)$ has $k - 1$ changes of sign for $-\infty < \omega < \infty$ and $dI/d\omega$ is negative in the neighborhood of $\omega = 0$.*

It has been shown formerly [2; theorem 2] that all subclass k functions are class k functions. Even with these more general definitions, the previously given proof applies in precisely the same way. Furthermore, all the theorems in Part II of Ref. [2] and in Ref. [3] continue to hold since their proofs did not make use of the rationality of $Z(s)$. (Theorem 6 of Ref. [3] must be modified slightly if it is to apply in this case.)

One particular result [2; theorem 9] that the unit impulse response corresponding to any class k function must satisfy is

$$| W(t) | \leq \frac{Kt^{k-1}}{(k-1)!}. \tag{11}$$

The principal objective of this paper is to show that the following bounds exist on a number of the derivatives of those $W(t)$ corresponding to subclass k functions. In particular, for $k \geq 2$

$$| W^{(\mu)}(t) | \leq \frac{Kt^{k-\mu-1}}{(k-\mu-1)!} \tag{12}$$

where the integer μ is restricted by $0 \leq \mu \leq (k-1)/2$ if k is odd and by $0 \leq \mu \leq k/2$ if k is even. In developing this result, some new properties of the subclass k functions will be obtained.

2. The following lemmas will be needed.

Lemma 1. *Let $F(s)$ satisfy conditions* (a) *and* (b) *of Definition* 1. *Then, for k odd, $R(\omega)$ must have at least $k - 1$ changes of sign in $-\infty < \omega < \infty$ and, for k even, $I(\omega)$ must have at least $k - 1$ changes of sign in $-\infty < \omega < \infty$.*

The proof of this lemma has been given previously [2; lemma 3]. It applies in the case where $F(s)$ is not rational since the rational property was not invoked anywhere in its proof.

Lemma 2. *Let k be even and let $Z(s)$ be a subclass k function. Then $sZ(s)$ is a subclass $(k-1)$ function.*

Proof. By condition (c′), $I(\omega)$ has $k - 1$ changes of sign in $-\infty < \omega < \infty$. Since $I(\omega)$ is an odd function [2; lemma 2], one of these sign changes occurs at $\omega = 0$. Now the real part of $sZ(s)$ equals $-\omega I(\omega)$ and, consequently, it has $(k-2)$ sign changes in $-\infty < \omega < \infty$. Since $dI/d\omega$ is negative in the neighborhood of $\omega = 0$, $-\omega I(\omega)$ is positive in the same neighborhood. Thus, $-\omega I(\omega)$ fulfills condition (c′). Conditions (a) and (b) are also fulfilled by $sZ(s)$ so that the lemma is established.

Theorem 1. *Let $k \geq 3$ and let $Z(s)$ be a subclass k function. Then,*

$$G(s) = \int_s^\infty s_0 Z(s_0)\, ds_0 \tag{13}$$

is a subclass $(k-2)$ function.

Proof. The theorem will first be established in the case where k is odd. Denoting the real part of $G(j\omega)$ by $R_G(\omega)$, we have

$$R_G(\omega) = -\int_\omega^\infty \omega_0 R(\omega_0)\, d\omega_0 \,. \tag{14}$$

Since $sZ(s)$ is analytic for $\sigma \geq 0$ and since $Z(s) = 0\,(1/s^k)$ as $s \to \infty$, (14) assumes the value of zero when the lower limit is set equal to $-\infty$. Hence,

$$R_G(\omega) = \int_{-\infty}^\omega \omega_0 R(\omega_0)\, d\omega_0 \,. \tag{15}$$

Integrating (15) by parts, the value of $R_G(\omega)$ at $\omega = 0$ may be expressed as

$$R_G(0) = -\int_{-\infty}^0 d\omega_1 \int_{-\infty}^{\omega_1} R(\omega_0)\, d\omega_0 \,.$$

Because $Z(s)$ is a subclass k function, $\int_{-\infty}^{\omega} R(\omega_0)\, d\omega_0$ will have $k - 2$ changes of sign in $-\infty < \omega < \infty$ one of which is at the origin. Hence,

$$\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\omega_1} R(\omega_0)\, d\omega_0 \tag{16}$$

will have $k - 3$ sign changes in $-\infty < \omega < \infty$. It follows that, since $R(\omega)$ is positive in the neighborhood of $\omega = 0$, (16) is negative at $\omega = 0$ so that $R_G(0)$ is positive.

Now consider the case where $k = 4\nu - 1$ ($\nu = 1, 2, 3, \cdots$). The quantity $\omega R(\omega)$ and, consequently, $R_G(\omega)$ become ultimately positive as ω decreases indefinitely toward $-\infty$. Hence, for $R_G(0)$ to be positive, $R_G(\omega)$ can have only an even number of sign changes in $-\infty < \omega < 0$. However, $\omega R(\omega)$ has $(k - 1)/2$ sign changes in $-\infty < \omega < 0$ and this is an odd number. Since integrating according to (15) can never increase the number of sign changes in $-\infty < \omega < 0$, $R_G(\omega)$ must have less than $(k - 1)/2$ sign changes in $-\infty < \omega < 0$. Moreover, by lemma 1, $R_G(\omega)$ has at least $(k - 3)/2$ sign changes in $-\infty < \omega < 0$. Thus, $R_G(\omega)$ has exactly $(k - 3)/2$ sign changes in $-\infty < \omega < 0$.

This result coupled with the facts that $R_G(\omega)$ is even and $R_G(0)$ is positive shows that $G(s)$ satisfies the requirements of condition (c′). Conditions (a) and (b) are also fulfilled so that $G(s)$ is a subclass $(k - 2)$ function.

A similar argument may be applied in the case where $k = 4\nu + 1$ ($\nu = 1, 2, 3, \cdots$).

Now let k be even. As before, it can be shown that $I_G(\omega)$ is given by

$$I_G(\omega) = \int_{-\infty}^{\omega} \omega_0 I(\omega_0)\, d\omega_0 \,. \tag{17}$$

This quantity is continuous and an odd function of ω. Hence, $I_G(0) = 0$. Also, its derivative is negative in the vicinity of $\omega = 0$ since $I(\omega)$ has this property and $I(0) = 0$.

Since $Z(s)$ is a subclass k function, $\omega I(\omega)$ has $(k - 2)/2$ changes of sign within the interval $-\infty < \omega < 0$. Let ω_i be a point where $I_G(\omega)$ changes sign. Since $I_G(\omega)$ is related to $\omega I(\omega)$ according to (17), $I_G(\omega)$ must have a smaller number of changes of sign in $-\infty < \omega < \omega_i$ than $\omega I(\omega)$ does. Furthermore, the point $\omega = 0$ is a point where $I_G(\omega)$ changes sign so that $I_G(\omega)$ has no more than $(k - 4)/2$ changes of sign in $-\infty < \omega < 0$. Hence, invoking Lemma 1, $I_G(\omega)$ has exactly this number of sign changes in $-\infty < \omega < 0$. It follows that $G(s)$ is a subclass $(k - 2)$ function. This completes the proof.

Theorem 2. Let $Z(s)$ be a subclass k function where $k \geq 2$. Also, let μ be any integer in the range $0 \leq \mu \leq (k - 1)/2$ if k is odd and let μ be any integer in the range $0 \leq \mu \leq k/2$ if k is even. Then $s^\mu Z(s)$ is a class $(k - \mu)$ function.

Proof. If a function $F(s)$ is a class k function, then the quantity $(-1)^h d^h F/ds^h$ is a class $(k + h)$ function. This result is an immediate consequence of the definition of a class k function.

Let k be odd ($k = 2\nu + 1$; $\nu = 1, 2, 3, \cdots$). To establish the conclusion, it will be shown that the function,

$$\int_{s}^{\infty} ds_{\mu-1} \int_{s_{\mu-1}}^{\infty} ds_{\mu-2} \cdots \int_{s_1}^{\infty} s_0^\mu Z(s_0)\, ds_0 \,, \tag{18}$$

is a class $(k - 2\mu)$ function where $0 \leq \mu \leq \nu = (k - 1)/2$.

Consider the innermost integral of (18). Integrating by parts repeatedly we may write

$$\begin{aligned}\int_s^\infty s_0^\mu Z(s_0)\,ds_0 &= s^{\mu-1}\int_s^\infty s_0 Z(s_0)\,ds_0 + (\mu-1)\int_s^\infty s_1^{\mu-2}\,ds_1\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\\ &= s^{\mu-1}\int_s^\infty s_0 Z(s_0)\,ds_0\\ &\quad + (\mu-1)s^{\mu-3}\int_s^\infty s_1\,ds_1\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\\ &\quad + (\mu-1)(\mu-3)\int_s^\infty s_2^{\mu-4}\,ds_2\int_{s_2}^\infty s_1\,ds_1\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\,.\end{aligned}$$

Continuing this process of integrating by parts the last term in this sum, the following result may be obtained. For μ odd,

$$\begin{aligned}\int_s^\infty s_0^\mu Z(s_0)\,ds_0 &= s^{\mu-1}\int_s^\infty s_0 Z(s_0)\,ds_0 + \cdots\\ &\quad + (\mu-1)(\mu-3)\cdots 4\cdot 2\int_s^\infty s_{(\mu-1)/2}\,ds_{(\mu-1)/2}\cdots\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\end{aligned} \tag{19}$$

and, for μ even,

$$\begin{aligned}\int_s^\infty s_0^\mu Z(s_0)\,ds_0 &= s^{\mu-1}\int_s^\infty s_0 Z(s_0)\,ds_0 + \cdots\\ &\quad + (\mu-1)(\mu-3)\cdots 3\cdot 1\cdot\int_s^\infty ds_{\mu/2}\int_{s_{\mu/2}}^\infty s_{\mu/2-1}\,ds_{\mu/2-1}\cdots\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\,.\end{aligned} \tag{20}$$

Repeatedly integrating in the above manner, (18) may be rearranged into the following finite sum.

$$\begin{aligned}&\int_s^\infty ds_{\mu-1}\int_{s_{\mu-1}}^\infty ds_{\mu-2}\cdots\int_{s_1}^\infty s_0^\mu Z(s_0)\,ds_0\\ &= \int_s^\infty s_{\mu-1}\,ds_{\mu-1}\int_{s_{\mu-1}}^\infty s_{\mu-2}\,ds_{\mu-2}\cdots\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\\ &\quad + A_1\int_s^\infty ds_\mu\int_{s_\mu}^\infty ds_{\mu-1}\int_{s_{\mu-1}}^\infty s_{\mu-2}\,ds_{\mu-2}\cdots\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\\ &\quad + A_2\int_s^\infty ds_{\mu+1}\int_{s_{\mu+1}}^\infty ds_\mu\int_{s_\mu}^\infty ds_{\mu-1}\int_{s_{\mu-1}}^\infty ds_{\mu-2}\int_{s_{\mu-2}}^\infty s_{\mu-3}\,ds_{\mu-3}\cdots\int_{s_1}^\infty s_0 Z(s_0)\,ds_0\\ &\quad + \cdots\,.\end{aligned} \tag{21}$$

In this expression, all the A_i are positive integers.

Now Theorem 1 may be applied repeatedly to each term on the right hand side of (21) to show that each term is a subclass $(k - 2\mu)$ function. That is, working from the innermost integral outward, Theorem 1 indicates that, when the integrand possesses the factor s, each integration yields a function of some subclass and the order of the resulting subclass is reduced by two from the order of the subclass of the initial function. Moreover, when the integrand does not have the factor s, each integration produces a subclass function whose order is reduced by one. Consequently, each term on the right hand side of (21) is a subclass $(k - 2\mu)$ function and, hence, a class $(k - 2\mu)$ function.

Furthermore, if $F(s)$ and $G(s)$ are class k functions and if A and B are positive numbers, then it follows from the definition of the class k functions that $AF(s) + BG(s)$ is also a class k function. Therefore, (21) is a class $(k - 2\mu)$ function and, by the remarks in the first paragraph of this proof, $s^\mu Z(s)$ is a class $(k - \mu)$ function.

The same result may be established in the case when k is even. ($k = 2\nu$; $\nu = 2, 3, 4, \cdots$. Lemma 2 states the theorem in the case when $k = 2$.) The same procedure as used before may be applied to the expression,

$$\int_s^\infty ds_{\mu-2} \int_{s_{\mu-2}}^\infty ds_{\mu-3} \cdots \int_{s_1}^\infty s_0^\mu Z(s_0)\, ds_0 \,, \tag{22}$$

to show that it is a class $(k - 2\mu + 1)$ function where $0 \leq \mu \leq \nu = k/2$. In this case (22) is rearranged into the following finite sum.

$$\begin{aligned} s \int_s^\infty s_{\mu-2}\, ds_{\mu-2} \int_{s_{\mu-2}}^\infty s_{\mu-3}\, ds_{\mu-3} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0 \\ + B_1 \int_s^\infty ds_{\mu-1} \int_{s_{\mu-1}}^\infty s_{\mu-2}\, ds_{\mu-2} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0 \\ + B_2 \int_s^\infty ds_\mu \int_{s_\mu}^\infty ds_{\mu-1} \int_{s_{\mu-1}}^\infty ds_{\mu-2} \int_{s_{\mu-2}}^\infty s_{\mu-3}\, ds_{\mu-3} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0 \\ + \cdots . \end{aligned}$$

In this expression, the B_i are all positive integers. Moreover, each term is a subclass $(k - 2\mu + 1)$ function by the same argument as before. Hence, (22) is in class $(k - 2\mu + 1)$. Differentiating it $\mu - 1$ times and multiplying by $(-1)^{\mu-1}$ will produce a class $(k - \mu)$ function. This completes the proof.

3. The principal conclusion of this paper may now be stated.

Theorem 3. *Under the hypothesis of Theorem* 2,

$$| W^{(\mu)}(t) | \leq K \frac{t^{k-\mu-1}}{(k - \mu - 1)!}.$$

Proof. The transfer function $Z(s)$ is asymptotic to K/s^k $(k \geq 2)$ as $s \to \infty$. Hence, by the initial value theorem [5; theorem 15, p. 267], the corresponding unit impulse response $W(t)$ and its first $k - 2$ derivatives all have an initial value of zero. Thus, for the stated ranges of μ, $s^\mu Z(s)$ is the Laplace transform of $W^{(\mu)}(t)$ [5; p. 129].

Finally, it has been shown [2; theorem 9], that the unit impulse response corresponding to a class k function is bounded according to (11). Consequently, the conclusion follows immediately from Theorem 2.

References

1. A. H. Zemanian, *A property of Hurwitz polynomials and some associated inequalities for Fourier transforms*, Proc. Am. Math. Soc. **8**, No. 4, 716–723 (Aug., 1957)
2. A. H. Zemanian, *On transfer functions and transients*, Quart. Appl. Math. **16**, No. 3, 273–294 (Oct., 1958)
3. A. H. Zemanian, *Some properties of rational transfer functions and their Laplace transformations*, Quart. Appl. Math., **17**, 245–253 (Oct., 1959)
4. O. Brune, *Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency*, J. Math. and Phys. **10**, 191–236 (Oct., 1931)
5. M. F. Gardner and J. L. Barnes, *Transients in linear systems*, vol. 1, John Wiley and Sons, New York, 1942