THE RAYLEIGH-RITZ METHOD FOR DISSIPATIVE OR GYROSCOPIC SYSTEMS*

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1. Introduction. According to a well known principle of Raleigh, the natural frequencies of a conservative mechanical system can not be decreased when constraints are imposed on the system [1]. This principle is the basis for the Rayleigh-Ritz method for the approximate determination of the natural frequencies.

Rayleigh treated vibration about *static equilibrium*, but he did not consider vibration about *steady motion*. Thus, his analysis does not apply to vibration of machines with electric motors. The difficulty is that gyroscopic forces have a different nature from forces arising from a potential.

In a previous paper, the writer treated highly dissipative systems [2]. A relation was introduced termed "overdamping", which insures that the general solution of the equations of motion is a sum of exponential decays. The decay values of overdamped systems are analogous to the natural frequencies of conservative systems. It was shown that Rayleigh's principle applies to the decay values.

In this paper, a simple transformation is introduced, which converts the equations of motion for a conservative system into the equations of motion for an overdamped system. Moreover, the addition of gyroscopic forces does not change this correspondence. By means of this device, it is seen that Rayleigh's principle is valid for the natural frequencies of vibration of such conservative gyroscopic systems.

2. Vibrations of conservative systems about static equilibrium. An example of the type of system considered by Rayleigh, is the triple pendulum, shown in Fig. 1.

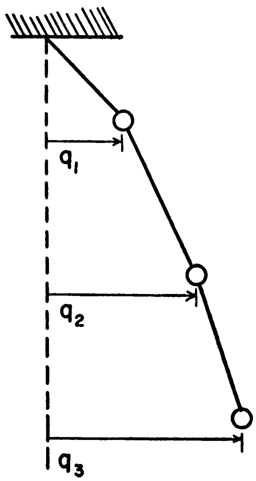

FIG. 1. Triple pendulum.

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The potential energy is of the form $c(q)/2$, where

$$c(q) = C_{11}q_1^2 + C_{22}q_2^2 + C_{33}q_3^2 + 2(C_{12}q_1q_2 + C_{23}q_2q_3 + C_{31}q_3q_1).$$

Of course, terms of higher order are neglected. The force components are obtained by differentiating the potential energy, so Newton's equations for small free vibrations are

$$A_i q_i'' + \sum_1^3 C_{ij}q_j = 0.$$

Here, A_1, A_2, and A_3, are the masses of the bobs, and a prime denotes differentiation with respect to the time, t.

In matrix notation, the equations of motion may be written as

$$Aq'' + Cq = 0. \tag{1}$$

In the general case of n degrees of freedom, q denotes a vector of n components, and A and C are real symmetric matrices with positive definite quadratic forms $a(q)$, and $c(q)$. Otherwise, A and C are arbitrary.

A normal mode of vibration is a solution of (1) of the form

$$q = ue^{i\omega t}. \tag{2}$$

Here, u is a constant vector, and the constant ω is termed a *natural frequency.* The equation which determines the normal modes of vibration is

$$-\omega^2 Au + Cu = 0. \tag{3}$$

A *constraint* is a linear homogeneous relation imposed on the coordinates such as

$$q_1 - 2q_2 + q_3 = 0. \tag{4}$$

For the triple pendulum, this constraint could be achieved physically by replacing the two strings connecting the bobs by a rigid rod.

To study the effect of constraints on the spectrum of natural frequencies, Rayleigh [1], introduced the functional

$$p^2(q) = \frac{c(q)}{a(q)}. \tag{5}$$

The stationary values of p are precisely the natural frequencies ω_i. The stationary values of p when q is restricted by (4) give the natural frequencies ω_i' of the constrained system. An additional constraint imposed leads to the frequencies ω_i'' of a doubly constrained system. The qualitative relationships are brought out in Fig. 2. Considering only the positive frequencies, we see that the constrained spectrum separates the original spectrum (This was proved independently by Routh, [3]).

3. Overdamped systems. The presence of dissipative forces can be accounted for by adding an appropriate term to (1). Then, the equation of motion becomes

$$Aq'' + Bq' + Cq = 0. \tag{6}$$

Here, B is a symmetric matrix whose quadratic form $b(q')$ gives the rate of dissipation of energy [1].

Attention is now focused on systems where the dissipation is large. This would be the case, for example, if the triple pendulum were immersed in a very viscous fluid.

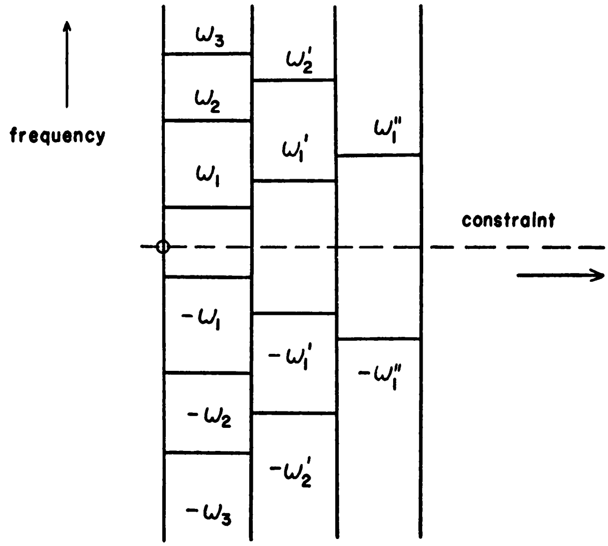

FIG. 2. Effect of constraints on the frequency spectrum.

The precise mathematical condition to be assumed is that the quadratic forms satisfy the following additional restriction

$$b^2(v) - 4a(v)c(v) > 0 \tag{7}$$

for an arbitrary real vector $v \neq 0$. In a previous paper, this was termed the *overdamping condition* [2]. It was shown that such onerdamped systens have *normal modes of decay.*

$$q = ue^{kt}. \tag{8}$$

Here, u is a constant vector, and the real constant k may be termed a *decay value.*

Substituting (8) in (6) gives

$$k^2Au + kBu + Cu = 0. \tag{9}$$

Forming the scalar product of (9) with u gives

$$\begin{aligned} k^2a + kb + c &= 0, \text{ and} \\ k &= [-b \pm(b^2 - 4ac)^{1/2}]/(2a) \end{aligned} \tag{10}$$

It was found that n of the normal modes require the positive choice of the radical in (10). These modes are termed *primary,* and their decay values are k_1 , k_2 , $\cdots$, k_n . Likewise, n normal modes require the negative sign. These modes are termed *secondary,* and their decay values are h_1 , h_2 , $\cdots$, h_n .

The following variational principle may be seen to apply to overdamped systems. Let the *primary functional* be defined as

$$p(v) = [-b(v) + \{b^2(v) - 4a(v)c(v)\}^{1/2}]/[2a(v)]. \tag{11}$$

Then the primary decay values are the stationary values of $p(v)$. Let the *secondary functional* be defined as

$$s(v) = [-b(v) - \{b^2(v) - 4a(v)c(v)\}^{1/2}]/[2a(v)]. \tag{12}$$

Likewise, the secondary decay values are the stationary values of $s(v)$.

A constraint on an overdamped system is seen to maintain condition (7). Thus the constrained system is also overdamped. The influence of constraint on the decay spectrum is illustrated in Fig. 3. A constraint is indicated by a prime. This diagram brings out the fact that the primary decay values of the constrained system separate the primary decay values of the original system. A similar separation relation holds for the secondary spectra.

It is seen from Fig. 3 that the following inequalities hold

$$h_1 \leq s(v) \leq h_3 < k_1 \leq p(v) \leq k_3 < 0. \tag{13}$$

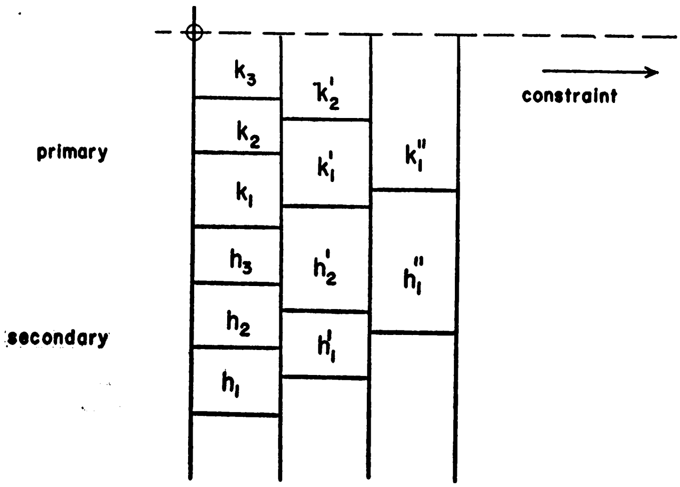

FIG. 3. Effect of constraints on the decay spectrum.

Here, v is an arbitrary vector; it results from a double constraint. Clearly, (13) would be useful for estimating the decay values of the overdamped triple pendulum.

These considerations show that the analogue of Rayleigh's principle holds for an overdamped system, provided the primary spectrum and the secondary spectrum are considered separately. Of course, even in the classical case, shown in Fig. 2, it is necessary to distinguish between the positive frequency spectrum and the negative frequency spectrum. This analogy is not accidental. Thus, by a trivial relaxation in the definitions, it is now to be shown that a conservative system is a special case of an overdamped system.

A *relative overdamped system* is defined by matrices A, B, and C, such that:

(i) They are symmetric.
(ii) The overdamping condition (7) holds.
(iii) A is positive definite.

With A, B, and C having these properties, let $k = k_0 + m$, in Eq. (9). Then (9) becomes $k_0^2 Au + k_0 B_0 u + C_0 u = 0$. Here $B_0 = 2mA + B$, and $C_0 = m^2 A + mB + C$. It may be verified that

$$b_0^2 - 4ac_0 = b^2 - 4ac > 0.$$

Moreover, if m is a sufficiently large positive constant, both B_0 and C_0 are positive definite. It follows that A, B_0, and C_0 define an "absolute" overdamped system. Thus, the spectrum of a relative overdamped system is simply the spectrum of an absolute overdamped system shifted upward.

The normal modes of vibration of a conservative system are defined by Eq. (3). But, clearly, the matrices A, 0, $-C$, correspond to a relative overdamped system. This proves that the conservative system is a special case.

4. Vibrations of conservative gyroscopic systems. Now of concern are mechanical systems whose small motions are governed by an equation of the form

$$Aq'' + Bq' + Cq = 0.$$

The matrices are to satisfy the conditions:

(i) A and C are symmetric positive definite.
(ii) B is antisymmetric.

We term such systems *regular*. It is seen that *regular* systems are conservative because the quadratic form $b(q')$ vanishes identically.

A simple example of a regular system is the gyroscopic pendulum shown in Fig. 4. A bob of mass m is spinning so as to have angular momentum L, directed along the rigid supporting rod of length h. The position of equilibrium is taken as origin of x, y coordinates in a horizontal plane.

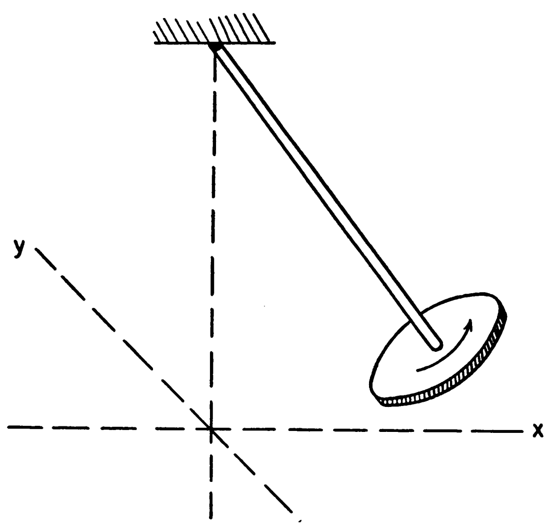

FIG. 4. Gyroscopic pendulum.

Taking moments about the point of support, leads to the following equations of motion.

$$mhx'' + Lh^{-1}y' + mgx = 0.$$

$$mhy'' - Lh^{-1}x' + mgy = 0.$$

It is easy to vertify that the normal modes of vibration are rotations in a circle with frequencies

$$\omega = \frac{L}{2mh^2} \pm \left[\left(\frac{L}{2mh^2} \right)^2 + \frac{g}{h} \right]^{1/2}.$$

The positive sign corresponds to counterclockwise rotation. A complete analysis of such pendulums is given by A. G. Webster [4]. General gyroscopic systems are treated by E. T. Whittaker [5].

Returning to the general case, it is seen that the equation for the normal modes of vibration of a regular system is

$$-\omega^2 Az + i\omega Bz + Cz = 0. \tag{14}$$

Here, the vector z may be complex, but it can be written as $z = u + iv$, where u and v are real vectors. This expression for z is substituted in (14), and the real and imaginary terms are separated. The two equations resulting, may be written as the following single equation, using compound matrices.

$$\left[\omega^2 \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \omega \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} - \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right] \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{15}$$

The matrices in (15) are symmetric. The overdamping condition is satisfied because

$$4(u \cdot Bv)^2 + 4[a(u) + a(v)][c(u) + c(v)] > 0.$$

Moreover, the first matrix in (15) is positive definite. Thus, (15) may be regarded as the equation for the normal modes of an overdamped system with $2n$ degrees of freedom.

If ω and (u, v) satisfy (15), then ω and $(v, -u)$ also satisfy (15). Thus, each decay value of (15) is "double". Otherwise, the decay spectrum of (15) is identical with the frequency spectrum of (14).

A constraint on (14), corresponds to two constraints on (15), one on u, and one on v. Or course, the constrained spectrum of (15) is double valued. Then it is seen that inequalities of the type (13) lead to the following separation property. The double values of the doubly constrained spectrum separate the double values of the original spectrum.

By the correspondence between the spectra of (14), and (15), it follows that the separation property holds for the natural frequencies of (14). Moreover, if ω is a natural frequency, it is seen from (14), that $-\omega$ is also. Thus, the frequency spectrum is exactly as shown in Fig. 2. This proves Rayleigh's principle for regular systems.

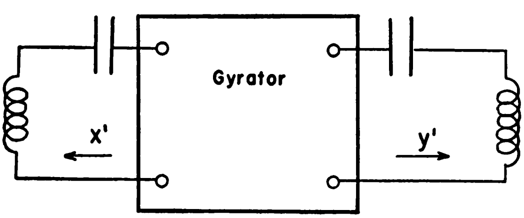

FIG. 5. Network analogue of the gyroscopic pendulum.

5. Non-reciprocal electrical networks. The classical analogy between mechanical systems and electrical networks makes force correspond to voltage, and velocity correspond to current. The matrices A, B, and C, correspond to matrices of inductance, resistance, and (capacitance)$^{-1}$. If A, B, and C, are symmetric, the network obeys the reciprocal law of Rayleigh [1].

A recent development in the network art has been networks which are the analogue of conservative gyroscopic systems. To aid in the analysis and synthesis of such non-reciprocal networks, Tellegen [6], introduced an ideal network element termed a gyrator. This concept is brought out in Fig. 5, which gives a network analogue of the gyroscopic pendulum.

It is seen that the gyrator serves as a representation of the antisymmetric B matrix.

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