on f itself can be relaxed, and the fluid may be more unstable. This instability in fact corresponds to the root zero of the characteristic equation

$$\tan R^{1/4} = \tanh R^{1/4}$$

given in the paper. The author is indebted to Mr. R. A. Wooding of Cavendish Laboratory of Cambridge for pointing out this most unstable mode.

ON THE SOMMERFELD HALF-PLANE PROBLEM*

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Abstract. A simple derivation of the Sommerfeld solution to the problem of the diffraction of a plane, scalar wave by a half-plane is given. The discusson is of interest mainly because of the simplicity of the argument; however, it is felt that the ansatz that forms the core of the argument is probably more generally applicable.

I. The method of deriving the Sommerfeld solution to the problem of the diffraction of a plane, scalar wave by a half-plane, presented here, became apparent during a study of the utility of conformal mapping in diffraction problems. The discussion is presented mainly because of the simplicity of the argument employed to deduce Sommerfeld's well-known result, but, beyond this, there are two other features of interest: first, one feels that the ansatz that constitutes the core of the argument is probably applicable more generally; and second, although the explicit conformal mapping used is trivial, the manner in which the mapping enters into the formulation of the boundary conditions in the ansatz may be suggestive in clarifying the relationship between conformal mapping and diffraction theory.

II. We seek a function, u, that is a solution of the wave equation[1]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u = 0, \tag{1}$$

corresponding to the incident wave[2]

$$\exp\left[-ikr\cos(\theta - \theta')\right], \tag{2}$$

and satisfying certain other conditions to be stated presently. These conditions, the wave equation, and the form of the incident wave, show that u is a function of $\theta - \theta'$ only; we may therefore allow θ' to approach zero and use

$$u_0 = \exp(-ikr\cos\theta) \tag{3}$$

as the incident wave, providing that, in the final result for u, we replace θ by $\theta - \theta'$ everywhere.

It is instructive to consider the problem simultaneously as it appears in Fig. 1, which shows the actual half-plane, coincident with the positive x-axis, and Fig. 2, which

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[1]We shall use the coordinates x, y; r, θ; and $z = y + iy$, $z^* = x - iy$, as convenience dictates.

[2]In contrast to the usual convention, note that this incident wave is traveling towards the x-axis, from above, at an angle θ'.

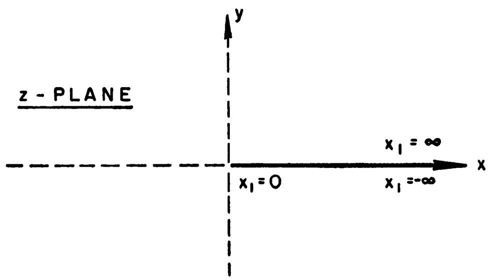

FIG. 1

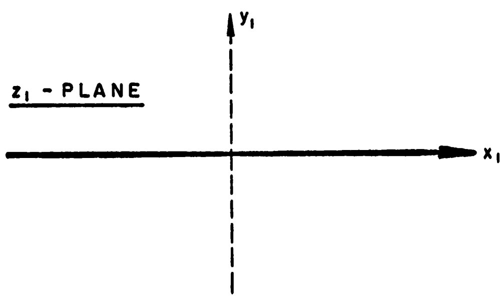

FIG. 2

shows the half-plane mapped on the entire real axis. (Figure 1 is related to Fig. 2 by $z = z_1^2$.)

We shall assume that θ' has approached zero from positive values of θ', so that the incident wave in Fig. 1 is regarded as coming on the half-plane from above, right, at grazing incidence, and moving in the negative x-direction; in Fig. 2, the same wave, suitably transformed, is incident on the entire real axis from above.

Suppose the boundary condition on the half-plane is $u = 0$; that is, u vanishes all along the real axis in the z_1 plane. This condition may be satisfied as follows. Assume we find a solution of the wave equation that has u_0 as its incident wave and that satisfies *any* boundary condition on the x_1-axis. From this solution we subtract the function obtained by reflecting the incident wave in the x_1-axis; the result is a solution of the wave equation that has the correct source above the x_1-axis and reduces to zero on the x_1-axis.

Thus, the only condition we need impose on u, other than that it is a solution of the wave equation, is that the wave u_0 is incident on the half-plane from above; the boundary condition on the half-plane may be chosen at our convenience.

One simple choice regards the half-plane as "black". That is, the half-plane absorbs the incident wave completely. No scattered wave is required, except at the edge of the half-plane; here, a scattered wave is needed in order to maintain the continuity of the wave function u in going around the half-plane.

This boundary condition means that we must find a (continuous) function u such that:

$$u \text{ satisfies the wave equation.} \tag{A-1}$$

$$u \to u_0 \text{ as } x_1 \to \infty \text{ for } y_1 > 0. \tag{A-2}$$

$$u \to 0 \text{ as } x_1 \to -\infty \text{ for } y_1 > 0. \tag{A-3}$$

We now proceed to satisfy these conditions. Our method is based on the fundamental ansatz:

$$u = u_0 I(x_1), \tag{4}$$

where we require that

$$I(\infty) = 1, \tag{5}$$

and

$$I(-\infty) = 0, \tag{6}$$

a pair of conditions that can be met by putting

$$I(x_1) = \int_{-\infty}^{ax_1} f(\tau)\, d\tau, \tag{7}$$

where a is an arbitrary constant.

(A-2) and (A-3) are now satisfied. Moreover, $f(\tau)$ is still arbitrary, leaving the possibility that it can be chosen to satisfy (A-1).

Since

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{|z_1|^2} \frac{\partial^2}{\partial z_1\, \partial z_1^*},$$

(A-1) demands that

$$\left\{\frac{1}{|z_1|^2} \frac{\partial^2}{\partial z_1\, \partial z_1^*} + k^2\right\} \exp\left[-\frac{ik}{2}(z_1^2 + z_1^{*2})\right] I\left(\frac{a}{2}(z_1 + z_1^*)\right) = 0. \tag{8}$$

From this, we deduce that $\left[f'(\tau) \equiv \frac{d}{d\tau} f(\tau)\right]$

$$\frac{f'(ax_1)}{f(ax_1)} = \frac{4ik}{a^2} \cdot ax_1\ ;$$

or,

$$f(\tau) = \alpha \exp\,(i2k\tau^2/a^2). \tag{9}$$

Putting this in (7), adjusting α to satisfy the normalization requirement (5), and using $x_1 = \rho^{1/2} \cos \theta/2$, we obtain:

$$I(x_1) = \int_{-\infty}^{a\rho^{1/2}\cos\theta/2} \left(\frac{2k}{\pi a^2 i}\right)^{1/2} \exp\,(2ik\tau^2/a^2) = (\pi i)^{-1/2} \int_{-\infty}^{(2k\rho)^{1/2}\cos\theta/2} \exp\,(i\tau^2)\, dt. \tag{10}$$

The complete solution is now given by making the replacement $\theta \to \theta - \theta'$ in (3), (4), and (10):

$$u(\theta;\, \theta') = (\pi i)^{-1/2} \exp\,[-ikr \cos(\theta - \theta')] \int_{-\infty}^{(2k\rho)^{1/2}\cos(\theta-\theta')/2} \exp\,(i\tau^2)\, d\tau. \tag{11}$$

The solution corresponds to a "black" half-plane, but, following the prescription given earlier, it can easily be transcribed into the solution for a perfectly conducting half-plane ($u = 0$ on the half-plane): In the z_1-plane, reflecting the source in the real axis changes θ_1' to $-\theta_1'$; this transformation replaces θ' by $-\theta'$ in the z-plane. Therefore, the required function is

$$u(\theta;\, \theta') - u(\theta;\, -\theta'), \tag{12}$$

since this takes on the value zero for $\theta = 0;\, 2\pi$.