ON THE NON-EXISTENCE OF TRANSONIC PERTURBATIONS*

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M. Schäfer [2] has developed a practical method to construct plane transonic potential flows directly by introducing a mesh of derived characteristics so defined that the mesh is real in both the supersonic and subsonic regions and joins continuously on the sonic line. Since the general solution of the equations for derived characteristics leads to great mathematical complications, Schäfer considers a simple solution which leads to the well-known Chaplygin solutions of the gasdynamic equations. These are a one parameter family of solutions which by suitable choice of the parameter, can be made to describe flows past profiles of practical interest.

Since the equations are linear in the derived characteristic variables which, in the case of the simple solution mentioned above, coincide with the polar hodograph variables, it is possible to find new solutions by superimposing solutions with different values of the parameter. In a second paper Schäfer [3] discusses the perturbation of profiles corresponding to special values of the parameter by superimposing solutions with positive integral values of the parameter on these specially chosen solutions which may be regarded as basic flows. The superimposed solutions are multiplied by coefficients which can be arbitrarily chosen to satisfy perturbed boundary conditions. Schäfer studies perturbations of the boundary conditions which are given by analytic (exponential) functions but which are practically localized in the supersonic region.

C. S. Morawetz [1] on the other hand, shewed that no perturbations involving only a finite segment of the supersonic boundary of a plane potential flow exist. If the perturbations studied by Schäfer could be regarded as numerical approximations of strictly localized perturbations, we would indeed have a family of flows depending—not necessarily smoothly[1]—on a parameter and with the same velocity at infinity namely, zero, which differ only along a finite segment of the supersonic boundary. Moreover, the theorems of Morawetz cannot be applied directly to these flows since in general a direction singularity occurs at infinity and hence infinity cannot be represented by a point in the hodograph plane considered by Morawetz.

Firstly, we remark that if the superposition is truncated, the perturbations must have an analytic character since each term in the summation is an analytic function of its variables. This means that singularities of curvature are impossible and the perturbations cannot be restricted to a finite segment of the supersonic boundary. We shew that the coefficients in the infinite series cannot be so chosen that a singularity is introduced into the boundary.

Using the simple solution of the equations for derived characteristics mentioned above, Schäfer [2] obtained the one parameter family of solutions of the gasdynamic equations in terms of certain metric functions as dependent variables and the polar

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[1]A set of flows given by infinite series of the solutions with arbitrary coefficients may be constructed by summing over the parameter.

hodograph variables v and ϑ as independent variables, known as the Chaplygin solutions. In Schäfer's notation the solutions are [2]:

$$\Xi = \frac{k}{\rho} v^{k-1} H_k(v^2) \begin{cases} \sin k\vartheta & \text{or} \\ \cos k\vartheta \end{cases}, \tag{1a}$$

$$\mathrm{H} = \frac{1}{\rho} [k v^{k-1} H_k(v^2) + 2 v^{k+1} H_k'(v^2)] \begin{cases} \cos k\vartheta & \text{or} \\ \sin k\vartheta \end{cases}, \tag{1b}$$

$$x - x_0 = \frac{1}{\rho}\left[\frac{v^{k+1}}{k+1} H_k' \sin (k+1)\vartheta + \frac{v^{k-1}}{k-1} (kH_k + v^2 H_k') \sin (k-1)\vartheta\right], \tag{2a}$$

$$y - y_0 = \frac{1}{\rho}\left[\frac{-v^{k+1}}{k+1} H_k' \cos (k+1)\vartheta + \frac{v^{k-1}}{k-1} (kH_k + v^2 H_k') \cos (k-1)\vartheta\right], \tag{2b}$$

$$\psi = v^k H_k(v^2) \begin{cases} \cos k\vartheta & \text{or} \\ \sin k\vartheta \end{cases}, \tag{3}$$

where

$$H_k = F(\alpha, -\beta; \gamma; v^2), \text{ the hypergeometric function,}$$

and

$$H_k'(w) = \frac{dH_k(w)}{dw}.$$

Ξ and H are the metric functions giving the relation between the cartesian coordinates x and y in the field of flow, and the derived characteristics which in the case considered coincide with the polar hodograph variables v and ϑ (v is the absolute value of the velocity vector made dimensionless with the velocity at zero pressure). ψ is the stream function, ρ the density made dimensionless with the stagnation density and κ the adiabatic gas constant.

We shall require an asymptotic expression for the hypergeometric function, $H_k(v^2)$, for large positive values of k. We write

$$H_k = \sum_{i=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\gamma+i)} \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)} \frac{\Gamma(i-\beta)}{\Gamma(-\beta)} \frac{v^{2i}}{i!}, \tag{4}$$

where

$$\alpha = \frac{1}{2}\left\{k - \frac{1}{\kappa-1} + \left[k^2 \frac{\kappa+1}{\kappa-1} + \frac{1}{(\kappa-1)^2}\right]^{1/2}\right\},$$

$$-\beta = \frac{1}{2}\left\{k - \frac{1}{\kappa-1} - \left[k^2 \frac{\kappa+1}{\kappa-1} + \frac{1}{(\kappa-1)^2}\right]^{1/2}\right\}.$$

and $\gamma = k + 1$. $\alpha > 0$ and $\beta > 0$ for positive k. We have for large k

$$\frac{\Gamma(\gamma)}{\Gamma(\gamma+i)} \sim \gamma^{-i}; \qquad \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)} \sim \alpha^i$$

and

$$H_k(v^2) \sim \sum_{i=0}^{\infty} \frac{\Gamma(i-\beta)}{\Gamma(-\beta)} \frac{1}{i!}\left[\frac{1}{2} v^2\left(1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right)\right]^i = \left[1 - \frac{1}{2} v^2\left(1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right)\right]^{\beta}. \tag{5}$$

By suitable choices of k Schäfer makes the solutions (1) represent basic flows of practical interest. Perturbations are then applied to these basic flows by putting

$$\Xi(v, \vartheta) = \Xi^*(v, \vartheta) + \frac{1}{\rho} \sum_{k=k_1}^{\infty} v^{k-1} H_k(v^2)(a_k \cos k\vartheta - b_k \sin k\vartheta), \tag{6a}$$

$$\mathrm{H}(v, \vartheta) = \mathrm{H}^*(v, \vartheta) + \frac{1}{\rho} \sum_{k=k_1}^{\infty} (kv^{k-1}H_k + 2v^{k+1}H_k')(a_k \sin k\vartheta + b_k \cos k\vartheta). \tag{6b}$$

The symbols with stars denote quantities in the undisturbed flow and plain symbols quantities in the disturbed flow. Since the equations for Ξ and H are linear, the perturbed metric functions as defined by (6) are again solutions of these equations. It is clearly more convenient to satisfy the perturbed boundary conditions in the physical plane and the expressions for x and y can be supplemented accordingly since they are linear combinations of Ξ and H. We shall now shew that the perturbation terms in (6) must be analytic functions of v and ϑ. We state the proposition in the following form: suppose the perturbation terms in (6) converge everywhere in the field of flow except possibly in a finite number of isolated points in the supersonic region. Then these terms converge absolutely and uniformly in all domains and they have term by term derivatives of all orders with respect to both variables.

Consider the function

$$v^{k-1}H_k(v^2). \tag{7}$$

This function tends to zero with increasing k, for in (5) $v < 1$ and β is asymptotically of the form

$$bk + O(k^{-1/2}), \qquad b > 0.$$

The asymptotic expression for (7) is

$$\left\{v\left[1 - \frac{1}{2}v^2\left(1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right)\right]^b\right\}^k = \{V(v)\}^k \tag{8}$$

with

$$b = \frac{1}{2}\left(\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2} - 1\right).$$

V increases with v in the interval

$$0 \leq v^2 < 2\left\{\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\left[1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right]\right\}^{-1}$$

In the latter point V has a maximum whereafter it decreases to the value zero for

$$v^2 = 2\left\{1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right\}^{-1} = v_2^2 .$$

We can therefore say that there is a limiting point of maxima of (7) at

$$v^2 = 2\left\{\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\left[1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right]\right\}^{-1} = v_1^2$$

with respect to positive k. Since the treatment of the perturbation term in (6) is the same whether we regard it as a sine series, a cosine series or a mixed series, we shall simplify by putting $b_k = 0$ for all k. We now put

$$a_k = f(k)A^k$$

and derive restrictions on A and f such that the conditions of our proposition are satisfied. A is simply a normalizing factor for the expression (8). Let A be chosen such that

$$Av_A\left[1 - \frac{1}{2}v_A^2\left(1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right)\right]^b = 1, \qquad v_A \neq v_1 .$$

With v_A in the range $0 \leq v_A < v_2$, the perturbation term in (6) would diverge as cosine series for all values of v in the interval $v_A < v \leq v_1$ or $v_1 \leq v < v_A$ according as $v_A < v_1$ or $v_1 < v_A$[1]. If the conditions of our proposition are still to be satisfied we must therefore have $v_A = v_1$. On $f(k)$ we must then obviously impose the restriction

$$f(k) = o(A^k).$$

The perturbation term can have at most one singular point with respect to v *viz.*, the point $v = v_1$. The order of this singularity is determined by the form of $f(k)$. It follows that the perturbation term is a uniformly and absolutely convergent cosine series with v in any subinterval of the interval

$$0 \leq v^2 \leq 2\left\{1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right\}^{-1} = v_2$$

which excludes $v = v_1$. We exclude the range

$$2\left\{1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right\}^{-1} < v^2 \leq 1$$

since in this range (8) becomes complex. We shall see that the mapping of the hodograph plane on the physical plane ceases to be unique at the point $v = v_2$.

We moreover know that each term of the cosine series is a uniformly and absolutely convergent power series of v in the range considered. Hence the perturbation term can be rearranged as a power series in v which has term by term derivatives of all orders with respect to v in all subintervals which do not include the point $v = v_1$.

Differentiating the perturbation term formally n times with respect to ϑ, we obtain

$$\Sigma(-)^t k^n f(k) A^k v^{k-1} H_k \begin{cases}\cos k\vartheta \\ \sin k\vartheta\end{cases} \quad \text{or, with} \quad t = [\tfrac{1}{2}n] + \tfrac{1}{2}.$$

But in this expression we still have

$$(-)^t k^n f(k) = o(A^k) \quad \text{since} \quad f(k) = o(A^k).$$

Hence the perturbation term also has term by term derivatives of all orders with respect to ϑ unless $v = v_1$. Note that if the perturbation term is singular at this value, then there will in general be a singular line in the supersonic region *viz.*, the isotach $v = v_1$.

We now shew that with the above restrictions on a_k our proposition also holds for (6b). The asymptotic expression for the coefficients of the perturbation term in (6b) is

[1]For we can find a k_0 for every such v such that $A^k v^{k-1} H_k \sim \eta^k$ for all $k > k_0$ with $\eta > 1$.

$$\left\{v\left[1-\frac{1}{2}v^2\left(1+\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right)\right]^b\right\}^k\left\{1-\frac{1}{2}\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\left[\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}+1\right]v^2\right\}. \tag{9}$$

The first factor is the same as for (6b) while the second factor has a zero in the point $v = v_1$ where $V(v)$ has a maximum. (9) therefore increases steadily until it approaches $v = v_1$ where it drops off sharply to negative values with the same amplitude in the range $v_1 < v < v_2$. The coefficients of (6b) are thus seen to have a limiting point of maxima in $v = v_1$ with respect to k. This limiting point is clearly never reached and away from this point the coefficients of (6b) become asymptotically

$$\left\{Av\left[1-\frac{1}{2}v^2\left[1+\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right]\right]^b\right\}^k$$

which is the same as the equivalent expression for (6a). Hence there can be no singularity in the perturbation term in (6b).

If we require furthermore that singularities only be propagated along characteristics, we must also exclude the singularity in (6a) since an isotach cannot coincide with a characteristic.

In order to exclude the range $v_2 < v < 1$ we consider the mapping of the hodograph plane on the physical plane. The Jacobian of this mapping is (see [3])

$$\frac{\partial(x, y)}{\partial(v, \vartheta)} = \frac{1}{v}\left(\frac{c^2 - v^2}{c^2}\,\Xi^2 + \mathrm{H}^2\right). \tag{10}$$

By comparing (10) with (8) and (9) we see that the expression

$$\left[1-\frac{1}{2}v^2\left(1+\left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right)\right]^b$$

is asymptotically a factor of the Jacobian so that the mapping will cease to be unique when $v = v_2$. Moreover a limiting line will generally have appeared in the basic flow before this velocity is reached—this is e.g., the case in the basic flow with $k = -\frac{5}{2}$ considered by Schäfer.

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