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THE MISES YIELD CONDITION FOR ROTATIONALLY SYMMETRIC SHELLS*

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Summary. The stress state in a rotationally symmetric shell under small displacements is characterized by the direct stresses and moments in the circumferential and longitudinal directions. If the shell material is perfectly plastic, it is desirable to express the material yield condition in terms of these stress resultants. Previous investigations have obtained this yield condition in certain special cases and for the maximum shear stress criterion in the general case.

Here, a derivation is given based on the Mises or octahedral-shear-stress criterion for both uniform and idealized-sandwich shells. The flow law relating extension and curvature rates of the middle surface to the stress state is also obtained. The general equations are obtained in closed parametric form and various special cases are explicitly presented.

1. Introduction. In the analysis of structures which are thin in one direction it is generally convenient to deal with stress resultants integrated over the thickness, rather than with the stresses themselves. The present paper will be directly concerned with rotationally symmetric shells. However, it will be shown that various simpler structures such as plates, slabs, arches, and frames may be regarded as suitable special cases of a theory based on rotationally symmetric shells. Although the methods used can theoretically be applied to still more general shell problems, the difficulties are formidable and the subject will not be treated here.

The state of stress in a symmetrically loaded rotationally symmetric shell is specified by circumferential and longitudinal direct stresses and moments and by a shear force. However, the basic assumptions are made that straight lines normal to the median surface of the shell remain straight and normal to the deformed median surface and that the displacements are small. It follows that shear strains are neglected so that the shear force is a reaction, not a generalized stress. Therefore, there are four generalized stresses, N_θ , N_ϕ , M_θ , M_ϕ . These quantities are related to the physical stress components by

$$N_\alpha = \int_{-H}^{H} \sigma_\alpha \, dZ, \qquad M_\alpha = \int_{-H}^{H} Z\sigma_\alpha \, dZ, \tag{1.1}$$

where the subscript α may stand for either θ or ϕ.

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As first shown by Prager [1] the generalized strain rates corresponding to these generalized stresses are the extension rates λ_θ, λ_ϕ and curvature rates K_θ, K_ϕ of the middle surface of the shell. In terms of these quantities, the strain rate components at a point are

$$\epsilon_\alpha = \lambda_\alpha + ZK_\alpha . \tag{1.2}$$

In order to formulate the shell problem, it is necessary to express all of the constituent equations in terms of the generalized stresses and strain rates. The equations of equilibrium and the relations between strain rates and velocities are easily derived by direct consideration of an infinitesimal element of the shell (see, for example [2]). In the case of an elastic material, the remaining equations necessary for a solution are obtained by combining Hooke's law with Eqs. (1.1) and (1.2) to obtain the elastic relations between generalized strain rate and stress.

For a perfectly plastic material, it is first necessary to express the yield condition in terms of generalized stresses. For various particular cases, such as beams under combined tension and bending [3] or circular plates [4], this is easily done for any yield condition. However, for the general rotationally symmetric shell, the problem becomes more difficult. Yield conditions have been obtained for shells where the material satisfies Tresca's yield condition of maximum shear [4-8]. In the present paper, we shall derive the yield condition for a shell whose material satisfies Mises yield condition of maximum octahedral stress.

Once the yield condition has been derived, the remaining equations are furnished by the plastic potential flow law. In geometrical terms, this states that the strain rate vector whose components are the generalized strain rates must be normal to the yield surface.

The present paper is concerned solely with the derivation of the yield condition and flow law for a rotationally symmetric shell. Applications of the theory will be reported on elsewhere [9].

2. Derivation of yield surface. As generalized stresses and strain rates for the rotationally symmetric shell we choose the dimensionless quantities

$$n_\alpha = N_\alpha/N_0 , \qquad m_\alpha = M_\alpha/M_0 , \qquad \lambda_\alpha , \qquad \kappa_\alpha = (M_0/N_0)K_\alpha \tag{2.1}$$

where

$$N_0 = 2H\sigma_0 , \qquad M_0 = H^2\sigma_0 \tag{2.2}$$

σ_0 being the yield stress and $2H$ the shell thickness. The dimensionless counterparts of Eqs. (1.1) and (1.2) are then

$$n_\alpha = (1/2) \int_{-1}^{1} s_\alpha \, dz, \qquad m_\alpha = \int_{-1}^{1} z s_\alpha \, dz$$
$$\epsilon_\alpha = \lambda_\alpha + 2z\kappa_\alpha , \qquad z = Z/H, \qquad s_\alpha = \sigma_\alpha/\sigma_0 . \tag{2.3}$$

At a plastic point the stresses must satisfy the Mises yield condition

$$s_1^2 - s_1 s_2 + s_2^2 = 1 \tag{2.4}$$

and the associated plastic potential flow rule. This latter states that the strain rate vector $(\epsilon_1 , \epsilon_2)$ is normal to the curve (2.4), hence

$$\epsilon_1 = \nu(2s_1 - s_2), \qquad \epsilon_2 = \nu(2s_2 - s_1), \tag{2.5}$$

where ν is an arbitrary positive scalar which describes the indeterminate magnitude of the strain rate vector.

By combining Eqs. (2.3)-(2.5) we can establish the generalized stresses as homogeneous functions of order zero of the generalized strain rates. It follows that the generalized stresses are thus dependent only upon three parameters representing the direction of the generalized strain rate vector in a four dimensional space. Since the representation of four variables in terms of three parameters is equivalent to a single equation relating these variables, we can thus obtain the desired yield condition.

There are, of course, many ways of choosing three parameters to represent the direction of the strain-rate vector; the following will prove convenient for later integration

$$\begin{aligned}(\lambda_1 , \lambda_2) &= \pm 2\nu t \cos (r + y \pm \pi/6),\\ (\kappa_1 , \kappa_2) &= \pm \nu \cos (r \pm \pi/6).\end{aligned} \tag{2.6}$$

Here, and in the following, the first and second terms in parenthesis are to be associated with the upper and lower signs, respectively.

In terms of r, t, and y, the stresses are

$$(s_1 , s_2) = -\frac{2}{3^{1/2}} \frac{t \sin (r + y \mp \pi/6) + z \sin (r \mp \pi/6)}{[(z + t \cos y)^2 + (t \sin y)^2]^{1/2}}. \tag{2.7}$$

Substitution of Eqs. (2.7) into Eqs. (2.3) and integration will give the desired yield condition. To carry out this integration we make the substitution defined in Fig. 1 and introduce new parameters p and q as the values of ω at $z = -1$ and $z = +1$, respectively. If t is finite and $t \sin y \neq 0$, then p and q must satisfy one or the other of

$$-\pi/2 < p \leq \omega \leq q < \pi/2, \tag{2.8a}$$

$$\pi/2 < p \leq \omega \leq q < 3\pi/2. \tag{2.8b}$$

In either case, the generalized stresses and strain rates are obtained in terms of p, q, and r [10]:

$$\begin{aligned}(n_1 , n_2)[3^{1/2} \sin (p - q)] &= \cos p \cos q \cos (r \mp \pi/6) \log \frac{(1 + \sin q)(1 - \sin p)}{(1 - \sin q)(1 + \sin p)}\\ &\quad + 2 \sin (r \mp \pi/6)(\cos p - \cos q)\end{aligned} \tag{2.9}$$

$$\begin{aligned}&(m_1 , m_2)[3^{1/2} \sin^2 (p - q)/2 \cos p \cos q)]\\ &\quad = [\sin (p + q) \cos (r \mp \pi/6) + \cos p \cos q \sin (r \mp \pi/6)]\\ &\qquad \cdot \log [(1 + \sin q)(1 - \sin p)(1 - \sin q)^{-1}(1 + \sin p)^{-1}]\\ &\qquad - 4 \cos (r \mp \pi/6)(\cos p - \cos q) - 2 \sin (r \mp \pi/6)(\sin q - \sin p)\end{aligned}$$

$$(\lambda_1 , \lambda_2) \sin (p - q) = \nu[\pm 2 \cos q \sin (r - p \pm \pi/6) \pm 2 \cos p \sin (r - q \pm \pi/6)]$$

$$(\kappa_1 , \kappa_2) = \pm \nu \cos (r \pm \pi/6) \sin (p - q).$$

Equations (2.9) give two different hypersurfaces, depending upon whether p and q satisfy Eq. (2.8a) or (2.8b). The dividing "hypercurve" between them may be found directly by setting $\sin y = 0$ before integrating the generalized stresses [10] or by letting

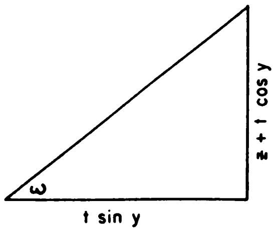

FIG. 1. Definition of ω

p and/or q tend to $\pm\pi/2$ in Eqs. (2.9). The result is

$$(n_1, n_2) = (2/3^{1/2})t \sin (r \mp \pi/6)$$
$$(m_1, m_2) = (2/3^{1/2})(1 - t^2) \sin (r \mp \pi/6) \tag{2.10}$$
$$-1 \leq t \leq 1.$$

Much simpler expressions are obtained for an ideal sandwich shell composed of two thin sheets of thickness J each, separated by a core of thickness $2H'$. The sheets are so thin that the stress variation across each sheet can be neglected; the core has no tensile strength but can carry the necessary shear. Evidently Eqs. (2.2) and the last Eq. (2.3) should be replaced by

$$N_0 = 2\sigma_0' J, \qquad M_0 = 2\sigma_0' H' J, \qquad s_\alpha = \sigma_\alpha/\sigma_0' . \tag{2.11}$$

It follows from statics that the stress resultants are related to the stresses s_α^+ and s_α^- in the top and bottom sheets, respectively, by

$$s_\alpha^+ = n_\alpha + m_\alpha \qquad s_\alpha^- = n_\alpha - m_\alpha . \tag{2.12}$$

The yield condition (2.4) cannot be violated by the stresses in either sheet. Therefore, substitution of Eqs. (2.12) in (2.4) shows that the yield condition consists of the two non-linear surfaces

$$(n_1 \pm m_1)^2 - (n_1 \pm m_1)(n_2 \pm m_2) + (n_2 \pm m_2)^2 = 1. \tag{2.13a, b}$$

The corresponding strain rates are

$$\lambda_1^\pm = \nu^\pm[2(n_1 \pm m_1) - (n_2 \pm m_2)], \qquad \lambda_2^\pm = \nu^\pm[2(n_2 \pm m_2) - (n_1 \pm m_1)]$$
$$\kappa_1^\pm = \pm\nu^\pm[2(n_1 \pm m_1) - (n_2 \pm m_2)], \quad \kappa_2^\pm = \pm\nu^\pm[2(n_2 \pm m_2) - (n_1 \pm m_1)]. \tag{2.14a, b}$$

If the stresses satisfy both of Eqs. (2.13), then the strain rates may be any combination of those in Eqs. (2.14a and b), provided only that the coefficients ν^+ and ν^- are both non-negative.

3. Special cases. Various structures of practical importance may be regarded as special cases of shells of revolution. Included in this category are arches, circular slabs under rotationally symmetric in-plane loads, rotational bending of a circular plate, and rotationally loaded circular cylindrical shells. In each case, the yield condition can be obtained as an appropriate special case of Eqs. (2.9) or (2.13). The following results of interest are then obtained.

a. Circular cylindrical shell. By assumption, $\kappa_\theta = 0$ hence $m_2 = m_\theta$ is a reaction

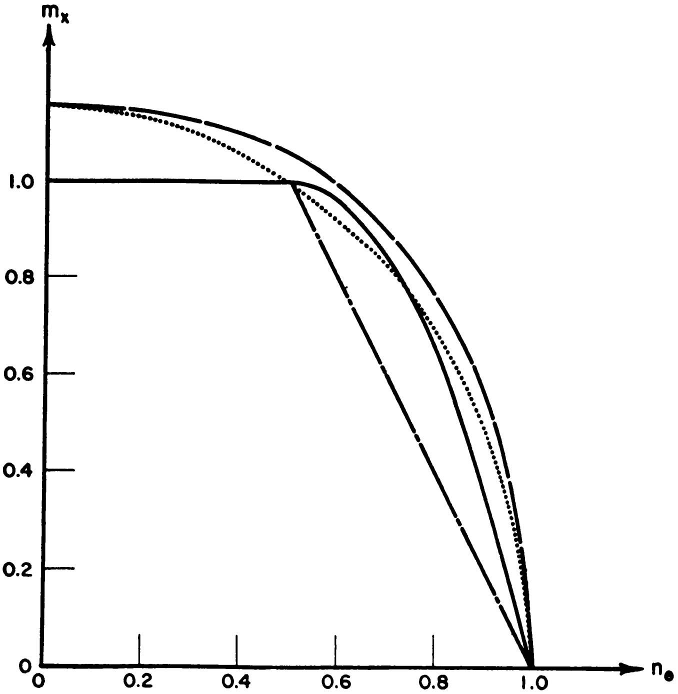

FIG. 2. Yield curves for circular cylindrical shell without end load.
— — —Mises condition, uniform shell
·········Mises condition, sandwich shell
————Tresca condition, uniform shell
—- - —Tresca condition, sandwich shell

to be eliminated from the yield condition. For the sandwich shell this process leads to two surfaces:

$$\pm 2(2n_\theta - n_x) = [4 - 3(n_x - m_x)^2]^{1/2} + [4 - 3(n_x + m_x)^2]^{1/2}. \tag{3.1}$$

For the uniform shell we obtain the two surfaces

$$\begin{aligned} n_x &= \pm \frac{2}{3^{1/2}} \frac{\cos p - \cos q}{\sin (p - q)}, \\ n_\theta &= \pm \left[-\frac{\cos p \cos q}{2 \sin (p - q)} \log \frac{(1 + \sin q)}{(1 - \sin q)} \frac{(1 - \sin p)}{(1 + \sin p)} + \frac{\cos p - \cos q}{3^{1/2} \sin (p - q)} \right], \\ m_x &= \pm \left[\frac{2 \cos^2 p \cos^2 q}{3^{1/2} \sin^2 (p - q)} \log \frac{(1 + \sin q)(1 - \sin p)}{(1 - \sin q)(1 + \sin p)} \right. \\ &\qquad \left. - 4 \frac{\cos p \cos q (\sin q - \sin p)}{3^{1/2} \sin^2 (p - q)} \right]. \end{aligned} \tag{3.2}$$

b. Circular cylindrical shell without end load. This case is a special case of the preceding one obtained by setting $n_x = 0$ in Eqs. (3.1) or (3.2). We obtain

$$n_\theta^2 + (3/4) m_x^2 = 1 \tag{3.3}$$

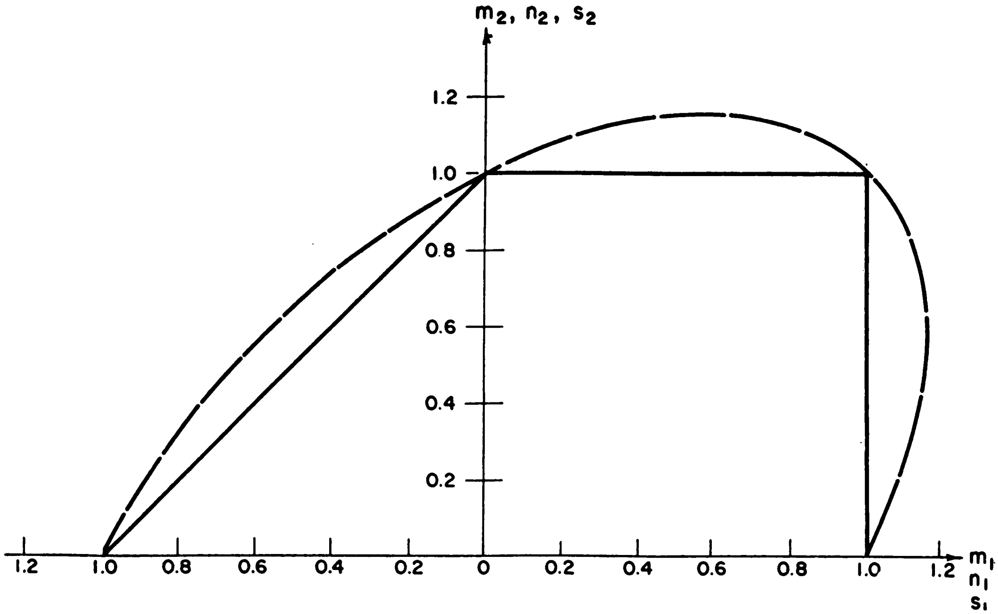

FIG. 3. Yield curves for stresses or circular plate under pure bending or pure tension
—— ——Mises condition, both shells
————Tresca condition, both shells

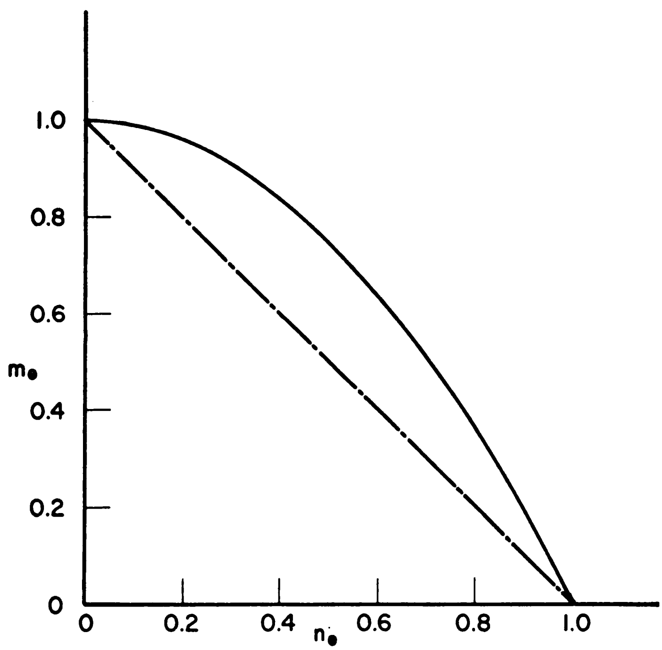

FIG. 4. Yield curves for arches
————Uniform shell, both conditions
—— - ——Sandwich shell, both conditions

for the sandwich shell. For the uniform shell

$$n_\theta = \pm(1/2) \cot q \log \frac{1 + \sin q}{1 - \sin q}, \qquad m_x = \pm\frac{1}{3^{1/2}}\left[\cot^2 q \log \frac{1 + \sin q}{1 - \sin q} - 2 \csc q\right]. \tag{3.4}$$

c. Circular slab under tension. Here $m_1 = m_2 = 0$ by assumption. The entire yield curve for the uniform shell is on the locus (2.10) of singular points. Setting $t = 1$ and eliminating r from the first line of (2.10) we obtain

$$n_1^2 - n_1 n_2 + n_2^2 = 1. \tag{3.5}$$

It follows from Eqs. (2.13) that (3.5) is also valid for the sandwich shell.

d. Circular plate under bending. In this case $n_1 = n_2 = 0$. Reasoning similar to that in case (*c*) shows that the yield curve for the uniform or sandwich plate is

$$m_1^2 - m_1 m_2 + m_2^2 = 1. \tag{3.6}$$

e. Curved beam or arch. Here transverse stresses are negligible, so that $n_\phi = m_\phi = 0$. Here also the singular curve for the uniform shell applies in the form

$$m_\theta = \pm(1 - n_\theta^2), \tag{3.7}$$

whereas the sandwich shell reduces to the linear expressions

$$n_\theta - m_\theta = \pm 1, \qquad n_\theta + m_\theta = \pm 1. \tag{3.8}$$

Figures 2, 3, and 4 show the special yield curves obtained in this section. For comparison, the corresponding curves for materials which satisfy the Tresca condition are also shown.

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