Equation (5) is an immediate consequence of (4), and the inequality (6) follows from (5) and (2). The first statement of the theorem is thus verified and the second statement becomes evident when $\Pi(t + \Delta t)$ is expanded in a Taylor's series in Δt about the time t.

If the complementary energy is defined by

$$\Pi_c \equiv \int_V \left(\int \epsilon_{ij} \, d\sigma_{ij} \right) dV - \int_{S_D} T_i u_i \tag{7}$$

and a functional W_c by

$$W_c(\sigma_0 , \epsilon_*) = \int_V \sigma^0_{ij} \epsilon^*_{ij} - \int_{S_D} T^0_i u^*_i \tag{8}$$

then the elastic principle of complementary energy states that Π_c is a minimum for the actual state among all statically admissible states and the analogous plastic principle states that

$$W_c(\sigma_0' , \epsilon_0') - W_c(\sigma', \epsilon') \geq 0. \tag{9}$$

Just as for the first principle one can easily prove two consequences of (9).

Theorem. Among all statically admissible rate states the actual rate state minimizes the time variations of

(*1*) *the complementary dissipation function* $W_c(\sigma', \epsilon)$:
(*2*) *the complementary energy* Π_c .

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ON THE SIMULTANEOUS DIAGONALIZATION OF TWO SEMI-DEFINITE MATRICES*

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1. Introduction. The use of congruency transformations for simultaneously diagonalizing two symmetric matrices, one of which is definite, is well known. One merely diagonalizes the definite matrix to (plus or minus) unity. This is then followed by an orthogonal transformation which diagonalizes the other matrix while preserving the unit matrix already obtained [1]. If, instead of being definite, one matrix is semi-definite,

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this method fails. However, if both matrices are semi-definite, this standard procedure can be extended.

2. Diagonalization. In the following let the superscript t denote matrix transposition and 1_r denote the unit matrix of order r. Further let the rth order zero matrix be denoted by 0_r. The main result is then the following theorem.

Theorem: Let A and B be nxn real, symmetric, positive semi-definite matrices. Then there exists a real non-singular matrix T and real diagonal matrices A_o and B_0, [see Eqs. (3) & (8)], such that

$$A = T^t A_o T$$
$$B = T^t B_o T \tag{1}$$

Proof: Let A have rank a and B rank b and assume that $b \geq a$. We first find a real, non-singular T_0 such that

$$A = T_o^t A_o T_o$$
$$B = T^t{}_o B' T_o \tag{2}$$

where

$$A_o = \text{diagonal } [1_a, 0_{n-a}]. \tag{3}$$

If any of the last $n - a$ diagonal elements of B' are zero, the corresponding entire row and column of B' are zero, since B' is semi-definite. For the last $n - a$ diagonal elements of B' which are nonzero, we can reduce the remaining nondiagonal elements in these rows and columns to zero. We must do this by always adding the diagonal element to the off diagonal element in order to preserve A_0. We can then write

$$A = T_o^t T_1^t A_o T_1 T_o$$
$$B = T_o^t T_1^t B'' T_1 T_o \tag{4}$$

where

$$B'' = \begin{bmatrix} B_a & & 0 \\ & 1_{b-\beta} & \\ 0 & & 0_{n-a-b+\beta} \end{bmatrix} \tag{5}$$

Here $\beta \geq 0$ is defined as the rank of B_a. We now diagonalize B_a by an orthogonal transformation T_a and put

$$T_2 = \begin{bmatrix} T_a & 0 \\ 0 & 1_{n-a} \end{bmatrix} \tag{6}$$

Now let

$$T = T_2 T_1 T_o. \tag{7}$$

Then Eq. (1) results with

$$B_o = \text{diagonal } [\lambda_1, \cdots, \lambda_\beta, 0_{a-\beta}, 1_{b-\beta}, 0_{n-a-b+\beta}] \tag{8}$$

where $\lambda_i > 0$, $i = 1, \cdots, \beta$.

By observing that neither the "sign" of A_0 nor that of B_0 enters into the proof, we see that we can diagonalize two semi-definite matrices (possibly of opposite sign). We can also easily extend the theorem to Hermitian matrices. Thus let a superscript asterisk denote complex conjugation and let A and B be complex Hermitian, positive semi-definite matrices. Proceeding as above, but using complex T_0 , T_1 and unitary T_a , we can write

$$\begin{aligned} A &= T^{t*} A_o T \\ B &= T^{t*} B_o T \end{aligned} \tag{9}$$

where A_0 and B_0 are as in Eqs. (3) and (8).

3. Applications. The above theorem is necessary for the synthesis of networks which are passive or active at a point (to be published, for the basic concepts see [2]). It also can be used to advantage in the synthesis of two element kind networks, as well as in studying equivalent networks (see pp. 96 and 142 of [3]). Its use in studying the vibrations of systems satisfying Lagrange's equations should also be apparent.

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AN UPPER BOUND ON RIGHT HALF PLANE ZEROS*

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Abstract. An upper bound is placed on the number of right half plane zeros of functions of the type $Z - m/n$. Z and m/n are RLC and LC driving point impedance functions respectively. In addition, it is shown that if $\text{Re}Z > 0$ on j axis, the number of right half plane zeros is determined precisely.

Introduction. In problems of control and network synthesis, it may be necessary to determine the number of right half plane zeros of certain impedance functions. In control problems, zeros in the right half plane may cause instability while in synthesis they may require active networks. In this paper an upper bound is placed on the number of these zeros of the class of functions $Z - m/n$ and $Z - n/m$. These terms are defined below.

Lemma.

Given: I. Z is prf (an RLC driving point impedance function).

II. $m + n$ is a Hurwitz polynomial, of degree d, of the complex variable S; m is an even and n is an odd function of S.

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